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**Abstract Wiener space approach to Hida calculus**

Shim, Youngsook Lee, Ph.D.

The Louisiana State University and Agricultural and Mechanical Col., 1987

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**ABSTRACT WIENER SPACE APPROACH TO HIDA CALCULUS**

**A Dissertation**

**Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
in partial fulfillment of the  
requirements for the degree of  
Doctor of Philosophy**

**in**

**The Department of Mathematics**

**by**

**Youngsook Lee Shim**

**B.S., The Seoul National University of Korea, 1974  
May 1987**

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## ABSTRACT

Let  $\mathcal{S}$  be the Schwartz space of rapidly decreasing real functions on  $\mathbb{R}$ . The dual space  $\mathcal{S}^*$  of  $\mathcal{S}$  consists of tempered distributions. The inclusion maps

$$\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^* \quad \text{are continuous.}$$

Hida's theory of Brownian and generalized Brownian functionals is the study of functionals defined on  $\mathcal{S}^*$ . In this dissertation, the triple  $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^*$  is replaced by an abstract Wiener space  $B^* \subset H \subset B$  and an abstract version of Hida's theory is developed. The Gaussian measure on  $\mathcal{S}^*$  in Hida's calculus is replaced by the standard Gaussian measure  $\mu$  on the space  $B$ . The  $\mathcal{S}^*$  valued curve  $\{\delta_t; t \in \mathbb{R}\}$  in Hida calculus is replaced by a  $B$ -valued curve  $\{O(t); t \in \mathbb{R}\}$ . The coordinate system, differential operator, and Laplacina operators with respect to  $\{\delta_t\}$  in Hida calculus can be defined with respect to  $\{O(t)\}$  in the Abstract Wiener space setup. Similar properties and theorems as in Hida calculus are obtained.



## INTRODUCTION

In 1975, Hida [3,5] initiated the study of Brownian functionals from the white noise point of view. This study leads to the theory of generalized Brownian functionals, which is referred nowadays as the Hida calculus. It is related to the curve  $\delta_t$ ,  $t \in \mathbb{R}$ , in the space  $\mathcal{D}'$  of tempered distributions. The purpose of this dissertation is to develop an abstract version of Hida calculus. Therefore, the space  $\mathcal{D}'$  is replaced by an abstract Wiener space  $B$  and  $\delta_t$  by a curve  $0_t$  in  $B$ .

The space  $\mathcal{D}'$  of tempered distributions is the dual space of the space  $\mathcal{D}$  of rapidly decreasing smooth functions on  $\mathbb{R}$ . Then we have the triple  $\mathcal{D} \subset L^2(\mathbb{R}) \subset \mathcal{D}'$ . There exists a unique probability measure  $\mu$  on  $\mathcal{D}'$  such that

$$\int_{\mathcal{D}'} e^{i\langle x, \xi \rangle} d\mu(x) = e^{-\frac{1}{2}\|\xi\|^2}, \text{ for } \xi \in \mathcal{D},$$

where  $\|\cdot\|$  is the  $L^2(\mathbb{R})$  norm. We call  $\mu$  the standard White noise measure on  $\mathcal{D}'$ . For each  $\xi$  in  $\mathcal{D}$ , the random variable  $\langle x, \xi \rangle$  on  $(\mathcal{D}', \mu)$  is normally distributed with mean 0 and variance  $\|\xi\|^2$ . Moreover, for  $f$  in  $L^2(\mathbb{R})$ , we can define  $\langle x, f \rangle$  to be the  $L^2(\mathcal{D}')$  limit of  $\langle x, f_n \rangle$ , where  $\{f_n\}$  is any sequence of  $\mathcal{D}$  such that  $\|f_n - f\| \rightarrow 0$ . We can easily see that  $\langle \cdot, f \rangle$  is a Gaussian random variable with mean 0 and variance  $\|f\|^2$ .

On the other hand, let  $(H, B)$  be an abstract Wiener space [11,12]. Then we have the triple  $B^* \subset H \subset B$ , where  $B^*$  is

the dual space of  $B$ . There exists a unique probability measure  $\mu$  on  $B$  such that

$$\int_B e^{i\langle x, z \rangle} d\mu(x) = e^{-\frac{1}{2}\|z\|^2}, \text{ for } z \in B^*$$

where  $\|\cdot\|$  denotes the  $H$ -norm,  $\mu$  is the standard Gaussian measure on  $B$ . For each  $z \in B^*$ , the random variable  $\langle x, z \rangle$  on  $(B, \mu)$  is defined everywhere and is normally distributed with mean 0 and variance  $\|z\|^2$ . If  $h \in H$ , then  $\langle x, h \rangle$  is defined as the  $L^2(\mu)$  limit of random variable  $\langle x, z_n \rangle$ , where  $\{z_n\}$  is any sequence in  $B^*$  such that  $\|z_n - h\| \rightarrow 0$ . The random variable  $\langle \cdot, h \rangle$  is normally distributed with mean 0 and variance  $\|h\|^2$ .

In Hida calculus we have the space  $(L^2)^+$  of test functionals and the space  $(L^2)^-$  of generalized Brownian functionals over the space  $\mathcal{A}^*$ , and  $(L^2)^+ \subset L^2(\mathcal{A}^*) \subset (L^2)^-$ . [3,5]. In Chapter 2 we will develop ordinary multiple integrals with respect to  $\mathcal{O}(t)$  for  $L^2(B)$  and define the space  $(L^2)^+$  of test random variables and the space  $(L^2)^-$  of generalized random variables by using the fact that  $L^2(B)$  has the orthogonal decomposition  $\bigoplus_{n=0}^{\infty} K_n$ , where  $K_n$  is the space of multiple Wiener integrals of order  $n$ .  $K_n$  can be identified with the space of  $n$ -fold symmetric tensor product of  $H$  [12]. In Hida calculus  $\{\dot{B}(t); t \in \mathbb{R}\}$  is taken as a coordinate system. It is a family of independent, identically distributed generalized random variables and takes the time propagation into account. By the same way we will take  $\langle \cdot, \mathcal{O}(t) \rangle; t \in \mathbb{R}$  as a coordinate system for our calculus. This system enjoys the same properties as  $\{\dot{B}(t); t \in \mathbb{R}\}$  in the Hida calculus.

One of the conditions assumed on  $\Theta(t)$  allows us to extend the H-inner product to  $B$  formally as

$$\langle x, y \rangle \equiv \int_{\mathbb{R}} \langle x, \Theta(t) \rangle \langle y, \Theta(t) \rangle dt, \quad x, y \in B.$$

Although it is only a formal expression, it plays a very important role in developing our calculus. In Chapter 3 we will define several operators such as  $\Theta(t)$ -differentiation operator  $\partial_{\Theta(t)}$ , the adjoint operator  $\partial_{\Theta(t)}^*$ ,  $\langle \cdot, \Theta(t) \rangle$  multiplication operator, and the various Laplacian operators. We will obtain similar results concerning these operators as in the Hida calculus. We will also define the Fourier transform in our setup. The connection between the Fourier transform and the Levy Laplacian will be studied in Chapter 4.

The following chart gives a comparison among the finite dimensional case, Hida calculus, and the abstract Wiener space setup of Hida calculus.

Finite dimensional case	Hida calculus	Abstract Wiener space
$\mathbb{R}^r$	$\mathcal{L}^*$	$B$
$\mathcal{L} \subset L^2(\mathbb{R}^r) \subset \mathcal{L}^*$	$\mathcal{L} \subset L^2(\mathbb{R}) \subset \mathcal{L}^*$	$(L^2)^+ \subset L^2(B) \subset (L^2)^-$
$(x_1, \dots, x_r)$	$\{\dot{B}(t); t \in \mathbb{R}\}$	$\{\langle \cdot, \Theta(t) \rangle; t \in \mathbb{R}\}$
$\partial_j = \partial/\partial x_j$	$\partial_t \equiv \dot{B}(t)$	$\partial_{\Theta(t)} \equiv \partial_{\langle \cdot, \Theta(t) \rangle}$
$x_j$	$\dot{B}(t)$	$\langle \cdot, \Theta(t) \rangle$
$\Delta$ (Laplacian)	$\Delta_G$ (Gross)	$\Delta_G$
	$\Delta_B$ (Beltrami)	$\Delta_B$
	$\Delta_L$ (Levy)	$\Delta_L$
	$\Delta_V$ (Volterra)	$\Delta_V$

## CHAPTER 1. Background

In this chapter we will give a brief review of the background for our work. Definitions and theorems are stated without proof and can be found in [3,5] and [11,12].

### §1. Rida's theory.

Let  $\mathcal{D}$  be the space of rapidly decreasing smooth real valued functions on  $\mathbb{R}$ . The dual space  $\mathcal{D}^*$  of  $\mathcal{D}$  consists of the tempered distributions. Thus we have the continuous inclusions

$\mathcal{D} \subset L^2(\mathbb{R}) \subset \mathcal{D}^*$ . The canonical bilinear form connecting  $\mathcal{D}$  and  $\mathcal{D}^*$  will be denoted by  $\langle x, \xi \rangle$ ,  $x \in \mathcal{D}^*$ ,  $\xi \in \mathcal{D}$ .

#### Theorem 1.1 (Bochner-Minlos theorem)

Let  $C(\xi)$  be a functional on  $\mathcal{D}$  satisfying (i) continuous; (ii) positive-definite, and; (iii)  $C(0) = 1$ . Then there exists a unique probability measure  $\mu$  on  $(\mathcal{D}^*, \mathcal{B})$  such that

$$C(\xi) = \int_{\mathcal{D}^*} \exp[i\langle x, \xi \rangle] d\mu(x)$$

where  $\mathcal{B}$  is the  $\sigma$ -field generated by the cylinder subsets of  $\mathcal{D}^*$  of the form  $\{x; (\langle x, \xi_1 \rangle, \dots, \langle x, \xi_n \rangle) \in B, B \text{ is a Borel subset of } \mathbb{R}^n, \xi_1, \dots, \xi_n \in \mathcal{D}, n = 1, 2, \dots\}$ .

**Definition 1.1** [3]

The measure space  $(\mathcal{A}^*, \mathcal{B}, \mu)$  determined by the characteristic functional  $C(\xi) = \exp[-\|\xi\|^2/2]$  is called a white noise space. Here  $\|\cdot\|$  is the  $L^2(\mathbb{R})$  norm.

**Remark:** By this definition, sometimes we call Hida calculus White Noise calculus. Moreover,

$$B(t, x) = \begin{cases} \langle x, 1_{(0, t]} \rangle, & t \geq 0 \\ -\langle x, 1_{(t, 0]} \rangle, & t < 0 \end{cases} \quad x \in \mathcal{A}^*$$

is a Brownian motion, and formally we have  $B(t) = \int_0^t x(u)du$  and  $\dot{B}(t) = x(t)$  where  $x \in \mathcal{A}^*$ . Therefore we will use  $\dot{B}$  for the element in  $\mathcal{A}^*$ . Then White Noise is the probability distribution of  $\{\dot{B}(t); t \in \mathbb{R}\}$ .

In distribution theory of finite dimensional space  $\mathbb{R}^n$ , we are concerned with a triple  $X \subset L^2(\mathbb{R}^n) \subset X^*$ , where  $X$  is the space of test functions and  $X^*$  is the space of generalized functions. Now we have the space  $L^2(\mathcal{A}^*)$  of Brownian functionals and Gaussian measure  $\mu$  as Theorem 1.1. So we will introduce the space  $X$  and  $X^*$  such that  $X \subset L^2(\mathcal{A}^*) \subset X^*$ .

**Theorem 1.2.** (Wiener-Ito decomposition theorem) [5]

$L^2(\mathcal{A}^*)$  has the direct orthogonal decomposition  $L^2(\mathcal{A}^*) = \bigcup_{n=0}^{\infty} K_n$  where  $K_n$  is the space of  $n$ -tuple Wiener integrals, i.e. each  $\phi$  in  $K_n$  has the following form

$$\phi(x) = \int_{\mathbb{R}^n} \dots \int f(u_1, \dots, u_n) dB(u_1, x) \dots dB(u_n, x)$$

where  $f \in L^2(\mathbb{R}^n)$ : the symmetric  $L^2(\mathbb{R}^n)$  function. Moreover;  $\|\phi\| = \sqrt{n!} \|f\|$ .

**Definition 1.2**

The S-transform on  $L^2(\mathcal{S}^*)$  is defined by

$$(S\phi)(\xi) = \int_{\mathcal{S}^*} \phi(x + \xi) d\mu(x).$$

**Theorem 1.3**

Suppose  $\phi \in K_n$  is of the form

$$\phi(x) = \int_{\mathbb{R}^n} \dots \int f(u_1, \dots, u_n) dB(u_1, x) \dots dB(u_n, x), \quad f \in \hat{L}^2(\mathbb{R}^n).$$

Then  $(S\phi)(\xi) = \int_{\mathbb{R}^n} \dots \int f(u_1, \dots, u_n) \xi(u_1) \dots \xi(u_n) du_1 \dots du_n$ .

**Remark:** From  $(S\phi)(\xi) = \int_{\mathbb{R}^n} \dots \int f(u_1, \dots, u_n) \xi(u_1) \dots \xi(u_n) du_1 \dots du_n$  we can see  $S\phi$  makes sense not only for  $f \in \hat{L}^2(\mathbb{R}^n)$ , but also for any  $f \in \mathcal{S}^*(\mathbb{R}^n)$ .

Noting the above remark we can consider following diagram

$$\begin{array}{ccccc} K_{n, a_n} & & K_n & & K_{n, -a_n} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \sqrt{n!} \hat{H}^{a_n}(\mathbb{R}^n) & \subset & \sqrt{n!} \hat{L}^2(\mathbb{R}^n) & \subset & \sqrt{n!} \hat{H}^{(-a_n)}(\mathbb{R}^n) \end{array}$$

where  $\hat{H}^a(\mathbb{R}^n)$  is the symmetric Sobolev space of order  $a > 0$ , i.e.

$$\hat{H}^a(\mathbb{R}^n) = \{f \in \mathcal{S}^* \mid \|f\|_a^2 = \int_{\mathbb{R}^n} (1 + |\lambda|^2)^a |\hat{f}(\lambda)|^2 d\lambda < \infty\}$$

where  $\hat{f}(\lambda)$  is the Fourier transform of  $f$ .

Let  $c_n$  be a bounded sequence of positive numbers. Then define

$$(L^2)_{a_n, c_n}^+ = \sum \frac{1}{c_n} K_{n, a_n}$$

$$(L^2)_{a_n, c_n}^- = \sum c_n K_{n, -a_n}.$$

Obviously we have

$$(L^2)_{a_n, c_n}^+ \subset (L^2(\mathcal{H}^*)) \subset (L^2)_{a_n, c_n}^-$$

Now  $(L^2)_{a_n, c_n}^+$  and  $(L^2)_{a_n, c_n}^-$  are called the space of test functionals and generalized functionals, respectively. Especially, in Hida calculus, we will assume that  $c_n = 1$  for every  $n$ ,  $a_n = \frac{n+1}{2}$  and  $K_{0, -a_0}$ , the nonstandard real number system.

### Definition 1.3

We define U-functional on  $(L^2)^-$  by

$$U(\phi)(\xi) = (S\phi)(\xi) \quad \text{for } \phi \in (L^2(\mathcal{H}^*)).$$

In general, we define  $U(\phi)(\xi) = \sum_{n=0}^{\infty} U_n(\xi)$  for  $\phi \in \sum_{n=0}^{\infty} \phi_n$  with  $\phi_n = \int \dots \int f(u_1 \dots u_n) dB(u_1) \dots dB(u_n)$  for  $f \in \hat{H}^{-\frac{n+1}{2}}(\mathbb{R}^n)$ , where

$$\text{where } U_n(\xi) = \int_{\mathbb{R}^n} f(u_1, \dots, u_n) \xi(u_1) \dots \xi(u_n) du_1 \dots du_n$$

For  $\phi \in (L^2)^-$  its U-functional is usually computed through renormalization. The procedure of renormalization is to find a sequence  $\phi_n$  in  $L^2(\mathcal{H}^*)$  such that  $\phi_n$  is Cauchy w.r.t.  $(L^2)^-$  norm. So U-functional of  $\phi$  is the limit of the U-functionals of  $\phi_n$ .

In Hida calculus,  $\{\dot{B}(t); t \in \mathbb{R}\}$  is taken to be a system of coordinates since  $\{\dot{B}(t); t \in \mathbb{R}\}$  is a family of independent random variables and often we have to consider time propagation. Therefore we need to define the coordinate differentiation  $\partial_t \equiv \partial_{\dot{B}(t)}$ .

### Definition 1.4

Let  $U$  be the U-functional of  $\phi \in (L^2)^-$ . Then the first variation of  $U$  is defined by

$$U(\xi + \delta\xi) = U(\xi) + \delta U + o(\delta\xi)$$

where

$$\delta U = \int_{\mathbb{R}} U'_\xi(u) \delta\xi(u) du.$$

If  $U'_\xi(t)$  is a  $U$ -functional of some function in  $(L^2)^-$ , then we define  $\partial_t \phi$  to be the generalized Brownian functional with  $U$  functional  $U'_\xi(t)$ , i.e.  $U(\partial_t \phi)(\xi) = U'_\xi(t)$ .

**Remark:** We have a formal expression  $\partial_t \dot{B}(u) = \delta_t(u)$  which is the analog of  $\partial_j x_k = \delta_{jk}$  in the finite dimensional case  $\mathbb{R}^n$ .

### Definition 1.5

1. The adjoint operator  $\partial_t^*$  is defined by

$$\langle \partial_t^* \phi, \psi \rangle = \langle \phi, \partial_t \psi \rangle \quad \text{for } \phi \in (L^2)^-, \psi \in (L^2)^+$$

2.  $\dot{B}(t)$  multiplication is defined by

$$\dot{B}(t)\phi \equiv \partial_t \phi + \partial_t^* \phi$$

The Laplacian operator and the Fourier transform will be studied in Chapter 3 and Chapter 4.

## **§2. Abstract Wiener space**

Let  $H$  be a real separable Hilbert space with norm  $\|\cdot\|$  and inner product  $(\cdot, \cdot)$ . Let  $\mathcal{P}$  denote the partially ordered set of finite dimensional orthogonal projections of  $H$ . A cylinder set  $E$  in  $H$  is a subset of  $H$  of the form  $E = \{x \in H; Px \in D, D \text{ is the Borel subset of } PH\}$ . Let us denote  $\mathcal{W}$  as the collection of the cylinder sets. Then  $\mathcal{W}$  is a field, but it is not a  $\sigma$ -field if  $\dim H = \infty$  [11].



**Definition 1.6.**

The Gauss measure  $\mu$  on  $H$  is the set function  $\mu: \mathcal{W} \rightarrow [0,1]$  defined by: for  $E = \{x \in H; Px \in D\}$ ,

$$\mu(E) = \left(\frac{1}{\sqrt{2\pi}}\right)^{\dim PH} \int_D e^{-\frac{\|x\|^2}{2}} dx,$$

where  $dx$  is the Lebesgue measure on  $PH$ .

**Lemma [11]**

$\mu$  is not  $\sigma$ -additive if  $\dim H = \infty$ .

**Definition 1.7**

A norm  $\|\cdot\|$  on  $H$  is called measurable if for any  $\epsilon > 0$ , there exists  $P_\epsilon \in \mathcal{P}$  such that

$$\mu\{x \in H; \|Px\| > \epsilon\} < \epsilon \quad P \in \mathcal{P} \text{ and } P \perp P_\epsilon.$$

**Remark:** 1.  $\|\cdot\|$  is not a measurable norm if  $\dim H = \infty$ .

2. There exists a constant  $c$  such that  $\|\cdot\| < c\|x\|$  for  $x \in H$ .

Let  $B$  be the completion of  $H$  w.r.t.  $\|\cdot\|$ .  $B$  is a separable real Banach space and  $H$  is dense in  $B$ .

**Definition 1.8**

The pair  $(H, B)$  is called an abstract Wiener space.

$H \subset B$  implies  $B^* \subset H^*$ . Therefore if we identify  $H^*$  with  $H$  by the Riesz's representation theorem, we have  $B^* \subset H^* \simeq H \subset B$ . Therefore,  $\langle z, h \rangle = (z, h)$  for all  $z \in B^*$ ,  $h \in H$ .

**Note:**  $\langle \cdot, \cdot \rangle$  denotes the dual pairing, and  $(\cdot, \cdot)$  denotes the inner product in  $H$ .

Now let  $\mathcal{W}_B$  denote the cylinder subsets of  $B$ , i.e.,  $\mathcal{W}_B$  is the collection of the set of the form

$$\{x \in B; (\langle x, z_1 \rangle, \dots, \langle x, z_n \rangle) \in D\}$$

where  $D$  is the Borel subset of  $\mathbb{R}^n$ , and  $z_1 \in B^*$ ,  $n = 1, 2, \dots$

Define  $\hat{\mu} : \mathcal{W}_B \rightarrow [0, 1]$  by

$$\hat{\mu}\{x \in B : (\langle x, z_1 \rangle, \dots, \langle x, z_n \rangle) \in D\} = \mu\{x \in H : ((x, z_1), \dots, (x, z_n)) \in D\}.$$

**Theorem 1.4** [11]

$\hat{\mu}$  has a unique  $\sigma$ -extension to the  $\sigma$ -field  $\sigma[\mathcal{W}_B]$  generated by  $\mathcal{W}_B$ .

**Note:** From now on, we will use  $\mu$  for the unique  $\sigma$ -extension of  $\hat{\mu}$  throughout this paper.

It is a well known fact that  $\langle x, z \rangle$ , for  $x \in B$ ,  $z \in B^*$ , is a random variable which is normally distributed with mean 0 and variance  $\|z\|^2$  w.r.t.  $\mu$ . Define a map  $n : H \rightarrow L^2(B, \mu)$  as follows: For  $z \in B^*$ ,  $n(z) = \langle \cdot, z \rangle$ . Since  $\|n(z)\|^2 = \|z\|^2$  we can extend  $n$  to  $H$  by continuity. So for  $h \in H$ ,  $\langle \cdot, h \rangle$  is defined a.e. ( $\mu$ ) in  $B$ . Obviously

$$\int_B e^{i\langle \cdot, h \rangle} d\mu(\cdot) = e^{-\frac{1}{2}\|h\|^2}.$$

Now for each  $x \in B$ , define  $\mu(x, D) = \mu(D - x)$

**Theorem 1.5** [12]

If  $h \in H$ , then  $\mu(h, \cdot)$  is absolutely continuous w.r.t.  $\mu(\cdot)$  and

$$\frac{d\mu(h, \cdot)}{d\mu(\cdot)}(x) = \exp\{\langle h, x \rangle - \frac{1}{2} \|h\|^2\}, \quad x \in B.$$

**Orthogonal decomposition of  $L^2(B, \mu)$** 

First recall that the Hermite polynomials  $H_n$ ,  $n = 0, 1, 2, \dots$  are defined by  $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ . They satisfy the recursion formula;

$$H_0(x) = 1$$

$$H_{n+1}(x) = 2x \cdot H_n(x) - 2n H_{n-1}(x).$$

Let  $\{e_k\}$  be a fixed orthonormal basis for  $H$ . For  $n_1 + \dots + n_r = n$  and  $i_1 < \dots < i_r$ , define a normalized Hermite polynomial

$$H_{i_1, \dots, i_r}^{n_1, \dots, n_r}$$

by

$$H_{i_1, \dots, i_r}^{n_1, \dots, n_r}(x) = \frac{1}{\sqrt{2^{n_1} \dots n_r!}} H_{n_1}\left(\frac{\langle x, e_{i_1} \rangle}{\sqrt{2}}\right) H_{n_2}\left(\frac{\langle x, e_{i_2} \rangle}{\sqrt{2}}\right) \dots H_{n_r}\left(\frac{\langle x, e_{i_r} \rangle}{\sqrt{2}}\right), \quad x \in B$$

**Theorem 1.6** [12]

$L^2(B, \mu) = \sum_{n=0}^{\infty} \oplus K_n$ , where  $K_n$  has the orthonormal basis

$$\{H_{i_1, \dots, i_r}^{n_1, \dots, n_r} ; n_1 + \dots + n_r = n, i_1 < \dots < i_r\}.$$

**Theorem 1.7** [12]

$K_n$  can be identified with the symmetric  $n$ -tensor product  $\bigotimes_n H$  of  $H$  by the following map:

$$H_{1_1, \dots, 1_r}^{n_1, \dots, n_r} \longrightarrow \frac{\sqrt{n!}}{\sqrt{n_1! \dots n_r!}} e_{1_1} \otimes \dots \otimes e_{1_r}.$$

## CHAPTER 2. The space $(L^2)^{-}$ of generalized random variables

### §1. Ordinary multiple Wiener integrals.

#### Definition 2.1

$\{\theta(t); t \in \mathbb{R}\}$  is called a  $B$ -valued curve in  $B$  if it satisfies the following conditions:

(i)  $\theta : t \rightarrow \theta(t) \in B$  is a continuous  $B$  valued function

(ii)  $\int_E \theta(u) du \in H$  for  $m(E) < \infty$  and  $\|\int_E \theta(u) du\|^2 = m(E)$

(iii)  $\int_{\mathbb{R}} \langle h, \theta(t) \rangle \langle k, \theta(t) \rangle dt = (h, k)$  for  $h, k \in H$ ,

where  $m$  is Lebesgue measure on  $\mathbb{R}$

Remark: From (iii) we can formally write

$$\langle x, \xi \rangle = \int_{\mathbb{R}} \langle x, \theta(t) \rangle \langle \xi, \theta(t) \rangle dt \quad \text{for } x \in B, \xi \in B^*.$$

Even more generally we will use formal representation

$$\langle x, y \rangle = \int_{\mathbb{R}} \langle x, \theta(t) \rangle \langle y, \theta(t) \rangle dt \quad \text{for } x, y \in B$$

#### Corollary 2.1

If  $E \cap E' = \emptyset$ , with  $m(E) < \infty$ ,  $m(E') < \infty$ , then

$$\left( \int_E \theta(u) du, \int_{E'} \theta(u) du \right) = m(E \cap E') = 0.$$

**Proof:** By the definition 2.1, (ii)

$$\left( \int_{E \cup E'} \Theta(u) du, \int_{E \cup E'} \Theta(u) du \right) = m(E \cup E') = m(E) + m(E') \\ \uparrow \\ \text{since } E \cap E' = \phi.$$

On the other hand

$$\begin{aligned} & \left( \int_{E \cup E'} \Theta(u) du, \int_{E \cup E'} \Theta(u) du \right) \\ &= \left( \int_E \Theta(u) du + \int_{E'} \Theta(u) du, \int_E \Theta(u) du + \int_{E'} \Theta(u) du \right) \\ &= \|\int_E \Theta(u) du\|_H^2 + \|\int_{E'} \Theta(u) du\|_H^2 + 2 \left( \int_E \Theta(u) du, \int_{E'} \Theta(u) du \right) \end{aligned}$$

Therefore we conclude that

$$\left( \int_E \Theta(u) du, \int_{E'} \Theta(u) du \right) = 0 \quad \text{when } E \cap E' = \phi. \quad (\text{Q.E.D.})$$

### **Corollary 2.2**

$$\left( \int_E \Theta(u) du, \int_{E'} \Theta(u) du \right) = m(E \cap E') \quad \text{if } m(E) < \infty, m(E') < \infty$$

**Proof:**

**Observation:**  $E = (E \setminus E') \cup (E \cap E')$  and  $E' = (E' \setminus E) \cup (E \cap E')$

$$\begin{aligned} & \left( \int_E \Theta(u) du, \int_{E'} \Theta(u) du \right) \\ &= \left( \int_{(E \setminus E') \cup (E \cap E')} \Theta(u) du, \int_{(E' \setminus E) \cup (E \cap E')} \Theta(u) du \right) \\ &= \left( \int_{E \setminus E'} \Theta(u) du + \int_{E \cap E'} \Theta(u) du, \int_{E' \setminus E} \Theta(u) du + \int_{E \cap E'} \Theta(u) du \right) \\ &= \left( \int_{E \setminus E'} \Theta(u) du, \int_{E' \setminus E} \Theta(u) du \right) + \left( \int_{E \setminus E'} \Theta(u) du, \int_{E \cap E'} \Theta(u) du \right) \\ &+ \left( \int_{E' \setminus E} \Theta(u) du, \int_{E \cap E'} \Theta(u) du \right) + \left( \int_{E \cap E'} \Theta(u) du, \int_{E \cap E'} \Theta(u) du \right) \\ &= \left( \int_{E \cap E'} \Theta(u) du, \int_{E \cap E'} \Theta(u) du \right) \\ &= m(E \cap E'). \end{aligned}$$

since

$$(E \setminus E') \cap (E' \setminus E) = \phi$$

$$(E \setminus E') \cap (E \cap E') = \phi$$

$$(E' \setminus E) \cap (E \cap E') = \phi.$$

(Q.E.D.)

From now on, we will assume that  $(H, B)$  satisfies the following conditions (i) and (ii):

(i) There exists a Hilbert space  $F \subset B$  such that  $H \subset F \subset B$  and  $(H, F)$  is also an abstract Wiener space.

(ii)  $\{\Theta(t); t \in \mathbb{R}\}$  is an  $F$  valued curve.

Since we assume that there exists  $F$  such that  $H \subset F \subset B$  and  $(H, F)$  is an abstract Wiener space, there exists an injective Hilbert Schmidt operator  $T$  on  $H$  such that  $(x, y)_F = (Tx, Ty)_H$  for  $x, y \in H$ . Then there exists an orthonormal basis  $\{e_n\} \subset H$  such that,  $Te_n = \lambda_n e_n$  with  $\sum \lambda_n^2 < \infty$  and  $e_n \in B^*$ , i.e.  $\lambda_n$ 's are eigenvalues of  $T$ ,  $e_n$ 's are eigenvectors of  $T$ . Obviously  $\|e_n\|_F = \lambda_n$ .

We can easily see that  $\{\xi_n = \frac{1}{\lambda_n} e_n, n = 1, 2, \dots\}$  is an orthonormal basis for  $F$ . Now for  $x \in F$ ,

$$\begin{aligned} x &= \sum_{n=1}^{\infty} (x, \xi_n)_F \xi_n \\ &= \sum_{n=1}^{\infty} (x, \frac{1}{\lambda_n} e_n)_F \frac{1}{\lambda_n} e_n \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n}\right)^2 (x, e_n)_F e_n \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n}\right)^2 \lambda_n^2 (x, e_n)_H e_n \\
&= \sum_{n=1}^{\infty} (x, e_n)_H e_n.
\end{aligned}$$

Therefore for fixed  $t \in \mathbb{R}$ ,  $\Theta(t)$  can be represented by

$\Theta(t) = \sum_{n=1}^{\infty} (\Theta(t), e_n)_H e_n$  even though  $\sum_{n=1}^{\infty} |(\Theta(t), e_n)_H|^2 = \infty$ . Here  $(\Theta(t), e_n)$  does not make sense, since  $\Theta(t) \in F \setminus H$ . But we will understand this  $\langle \Theta(t), e_n \rangle$  where  $\langle, \rangle$  means dual pairing between  $\Theta(t) \in B$  and  $e_n \in B^*$ .

**Note:**

$$\begin{aligned}
\|\Theta(t)\|_F^2 &= \sum_{n=1}^{\infty} |(\Theta(t), \xi_n)_F|^2 = \sum_{n=1}^{\infty} |(\Theta(t), \frac{1}{\lambda_n} e_n)_F|^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n}\right)^2 |(\Theta(t), e_n)_F|^2 = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n}\right)^2 |\lambda_n^2 (\Theta(t), e_n)_H|^2 \\
&= \sum_{n=1}^{\infty} \lambda_n^2 (\Theta(t), e_n)_H^2 < \infty
\end{aligned}$$

i.e., even though  $\sum_{n=1}^{\infty} |(\Theta(t), e_n)_H|^2 = \infty$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 |(\Theta(t), e_n)_H|^2 < \infty$  for each fixed  $t \in \mathbb{R}$ .

Now we can define ordinary multiple Wiener integrals, w.r.t.,  $\Theta(t)$  in Ito's sense [8]. Let's define  $\beta(E, x)$  as follows

$$\beta(E, x) = \langle x, \int_E \Theta(u) du \rangle \quad \text{for } E \in \mathcal{B}(\mathbb{R}), x \in B.$$

Then  $\beta(E, x)$  is a normal random measure by Corollary 2.2., i.e.,

$$\int_B \beta(E, x) \beta(E', x) d\mu(x) = m(E \cap E')$$

for any  $E, E'$  with  $m(E) < \infty$ ,  $m(E') < \infty$  where  $m$  is Lebesgue measure on  $\mathbb{R}$ .



Since Lebesgue measure satisfies the continuity condition we can define multiple Wiener integral in Ito's sense.

**Note:** Continuity condition. For any  $E \in \mathcal{B}(\mathbb{R})$  with  $m(E) < \infty$  and any  $\epsilon > 0$  there exists a decomposition of  $E$ :

$$E = \bigcup_{i=1}^n E_i \quad \text{such that} \quad m(E_i) < \epsilon, \quad i = 1, 2, \dots, n.$$

Now suppose  $f \in L^2(\mathbb{R}^p)$  is a special elementary function, i.e.,

$$f(t_1, \dots, t_p) \begin{cases} a_{i_1, \dots, i_p} & \text{for } (t_1, \dots, t_p) \in T_{i_1} \times \dots \times T_{i_p} \\ 0 & \text{elsewhere} \end{cases}$$

where  $T_1, T_2, \dots, T_n$  are disjoint and  $m(T_i) < \infty$ ,  $i = 1, 2, \dots, n$  and  $a_{i_1, \dots, i_p} = 0$  if any two of  $i_1, \dots, i_p$  are equal. Then define

$$\begin{aligned} I_p(f) &= \int \dots \int f(t_1, \dots, t_p) d\beta(t_1) \dots d\beta(t_p) \\ &= \sum a_{i_1, \dots, i_p} \beta(T_{i_1}) \dots \beta(T_{i_p}). \end{aligned}$$

We have the following properties

$$(1) \quad I_p(af + bg) = aI_p(f) + bI_p(g)$$

$$(2) \quad I_p(f) = I_p(\hat{f})$$

where  $\hat{f}$  is the symmetrization of  $f$ , i.e.,

$$\hat{f}(t_1, \dots, t_p) = \frac{1}{p!} \sum_{\pi} f(t_{\pi(1)} \dots t_{\pi(p)}),$$

where the summation runs over all permutations  $\pi$  of  $(1, 2, \dots, p)$ .

$$(3) \quad (I_p(f), I_p(g)) = p! (\hat{f}, \hat{g})$$

$$(4) \quad (I_p(f), I_q(g)) = 0 \quad \text{if } p \neq q.$$

**Note:** For the proof of the above properties, see the paper by Ito [8].

Ito showed also that this integral can be extended to the functions  $f$  in  $L^2(\mathbb{R}^n)$ .

**Theorem 2.1** [8]

Let  $\xi_1(t), \xi_2(t), \dots$  be an orthogonal system of real valued functions in  $L^2(\mathbb{R})$ . Then

$$\begin{aligned} & \int \dots \int \xi_1(t_1) \dots \xi_1(t_{p_1}) \xi_2(t_{p_1+1}) \dots \xi_2(t_{p_1+p_2}) \xi(t_{p_1+p_2+1}) \dots \\ & \dots \xi_n(t_{p_1+\dots+p_n}) d\beta(t_1) \dots d\beta(t_n) \\ &= \frac{\prod_{v=1}^n H_{p_v} \left( \frac{1}{\sqrt{2}} \int \xi_v(t) d\beta(t) \right)}{(\sqrt{2})^{p_v}}. \end{aligned}$$

**Theorem 2.2** [8]

For any  $f(\beta(E, x))$ ,  $E \in \mathcal{B}(\mathbb{R})$  with  $m(E) < \infty$  satisfying  $\int_B |f(\beta(E, x))|^2 d\mu(x) < \infty$ , it can be represented by

$$f(\beta(E, x)) = \sum_p \sum_{p_1+\dots+p_n=p} \sum_{a_1 \dots a_n} a_{p_1 \dots p_n}^{\alpha_1 \dots \alpha_n} \prod_{v=1}^n H_{p_v} \left( \frac{1}{\sqrt{2}} \int \xi_{\alpha_v}(t) d\beta(t) \right)$$

**Theorem 2.3**

If  $\phi \in K_n$ , then

$$\phi(x) = \int_{\mathbb{R}^n} f(u_1, \dots, u_n) d\beta(u_1, x) \dots d\beta(u_n, x), \quad f \in \hat{L}^2(\mathbb{R}^n),$$

$$f(u_1, \dots, u_n) \in \hat{L}^2(\mathbb{R}^n)$$

where  $\hat{L}^2(\mathbb{R}^n)$  is the space of symmetric functions in  $L^2(\mathbb{R}^n)$ .

Proof: We know  $k_n$  has the orthonormal basis of the form

$$\left\{ \prod_{i=1}^k \frac{1}{\sqrt{2^{n_{i_1}}! \dots n_{i_k}!}} H_{n_{i_1}} \left( \frac{\langle x, e_{i_1} \rangle}{\sqrt{2}} \right); \sum n_{i_1} = n, i_1 < i_2 < \dots \right\}$$

where  $\{e_{i_1}\}$  : ONB for  $H$ . Therefore it is enough to show that

$$\langle x, e_{i_1} \rangle = \int_{\mathbb{R}} \xi_{i_1}(t) d\beta(t, x)$$

by the Theorem 2.1. Also it is enough to show that

$$\int \xi_{i_1}(u) \Theta(u) du \in H \text{ and } \left\{ \int \xi_{i_1}(u) \Theta(u) du \right\}_{i_1=1}^{\infty} \text{ is ONB for } H$$

since

$$\int_{\mathbb{R}} 1_E(u) d\beta(u, x) = \beta(E, x) = \langle x, \int_E \Theta(u) du \rangle = \langle x, \int_{\mathbb{R}} 1_E(u) \Theta(u) du \rangle$$

Now define  $i : L^2(\mathbb{R}) \rightarrow H$  such that  $i(1_E) = \int_E \Theta(u) du$ . Since

$\|1_E\|_{L^2(\mathbb{R})}^2 = \|\int_E \Theta(u) du\|_H^2$ ,  $i$  is a densely defined isometry. Therefore we can naturally extend  $i$  to  $L^2(\mathbb{R})$ . Then

$$i(\xi_{i_1}) = \int_{\mathbb{R}} \xi_{i_1}(u) \Theta(u) du \in H \text{ for } \{\xi_{i_1}\} \text{ ONB for } L^2(\mathbb{R}) \text{ and}$$

$$\left( \int_{\mathbb{R}} \xi_{i_1}(u) \Theta(u) du, \int_{\mathbb{R}} \xi_{j_1}(u) \Theta(u) du \right)_H = (\xi_{i_1}(u), \xi_{j_1}(u))_{L^2(\mathbb{R})} = \delta_{i_1, j_1}$$

Therefore  $\left\{ \int_{\mathbb{R}} \xi_{i_1}(u) \Theta(u) du \right\}_{i_1=1}^{\infty}$  is ONB for  $H$ . So let

$i(\xi_{i_1}) = \int_{\mathbb{R}} \xi_{i_1}(u) \Theta(u) du \equiv e_{i_1}$  then we can fix  $\{e_{i_1}\}_{i_1=1}^{\infty}$ , ONB for  $H$ , for each  $\{\xi_{i_1}\}$  ONB for  $L^2(\mathbb{R})$ . (Q.E.D.)

**Note:** Symbolically we can write

$$\phi(x) = \int_{\mathbb{R}} f(u_1, \dots, u_n) d\beta(u_1, x) \dots d\beta(u_n, x)$$

$$= \int_{\mathbb{R}^n} f(u_1, \dots, u_n) : \langle x, \theta(u_1) \rangle \dots \langle x, \theta(u_n) \rangle : du_1 \dots du_n$$

for  $\phi \in K_n$ , where  $: : \dots :$  denotes the renormalization which we will define later.

### Example 1

We have  $\mathcal{A} \subset \mathcal{A}_p \subset L^2(\mathbb{R}) \subset \mathcal{A}_p^* \subset \mathcal{A}^*$  as before. In the Hida calculus,  $\theta(t) = \delta_t \in \mathcal{A}_p^*$  when  $p > 1$ , i.e.,  $\langle \theta(t), \xi \rangle = \langle \delta_t, \xi \rangle = \xi(t)$  where  $\xi \in \mathcal{A}$ . We show that  $\delta_t$  satisfies the conditions for  $\theta(t)$  in Definition 2.1. First we need to verify  $\delta_t : t \longrightarrow \delta_t \in \mathcal{A}_p^*$  is continuous. It is a well-known fact that  $\delta_t \in \mathcal{A}_{-1} = \mathcal{A}_1^*$ , so we need to show that  $\|\delta_{t+\Delta}(x) - \delta_t(x)\|_{H_{-1}} \rightarrow 0$  as  $\Delta \rightarrow 0$  where  $\|\cdot\|_{-1}$  is  $H_{-1}$  norm

$$\begin{aligned} \delta_{t+\Delta}(x) - \delta_t(x) &= \int_{\mathbb{R}} e^{-1xy} (\delta_{t+\Delta}(y) - \delta_t(y)) dy \\ &= e^{-1(t+\Delta)x} - e^{-1tx} \\ &= e^{-1tx} [e^{-1\Delta x} - 1]. \end{aligned}$$

Since  $|e^{-1\Delta x} - 1|^2 \leq 4$ , we can apply the Lebesgue Dominated Convergence theorem. Therefore

$$\begin{aligned} \|\delta_t(x) - \delta_{t+\Delta}(x)\|_{H_{-1}}^2 &= \int_{\mathbb{R}} \frac{1}{1+x^2} \cdot |\delta_{t+\Delta}(x) - \delta_t(x)|^2 dx \\ &= \int_{\mathbb{R}} \frac{1}{1+x^2} |e^{-1(t+\Delta)x} - e^{-1tx}|^2 dx \\ &= \int_{\mathbb{R}} \frac{1}{1+x^2} |(1 - e^{-1t\Delta})|^2 dx \\ &= \int_{\mathbb{R}} \frac{1}{1+x^2} |e^{-1\Delta x} - 1|^2 dx \end{aligned}$$

$$\begin{aligned} \lim_{\Delta \rightarrow 0} \|\delta_{t+\Delta}(x) - \delta_t(x)\|_{H^{-1}}^2 &= \lim_{\Delta \rightarrow 0} \int_{\mathbb{R}} \frac{1}{1+x^2} |e^{-i\Delta x} - 1|^2 dx \\ &= \int_{\mathbb{R}} \lim_{\Delta \rightarrow 0} \left[ \frac{1}{1+x^2} |(e^{-i\Delta x} - 1)|^2 \right] dx = 0. \end{aligned}$$

Next we need to check that  $\int_E \delta_t(u) du \in H = L^2(\mathbb{R})$  and

$\|\int_E \delta_t(u) du\|^2 = m(E)$  when  $m(E) < \infty$ . By the distribution theory

$\int_E \delta_t(u) du = \int_{\mathbb{R}} 1_E(u) \cdot \delta_t(u) du = 1_E(t)$  and  $1_E(t) \in L^2(\mathbb{R})$  when

$m(E) < \infty$ . Obviously  $\|1_E(t)\|_{L^2(\mathbb{R})}^2 = m(E)$ . Finally we need to check that

$$\int_{\mathbb{R}} \langle \delta_t, h \rangle \langle \delta_t, k \rangle dt = (h, k)$$

where  $h, k \in L^2(\mathbb{R})$  and  $(\cdot, \cdot)$  is inner product in  $L^2(\mathbb{R})$

$$\begin{aligned} \int_{\mathbb{R}} \langle \delta_t, h \rangle \langle \delta_t, k \rangle dt &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \delta_t(u) h(u) du \right) \left( \int_{\mathbb{R}} \delta_t(v) k(v) dv \right) dt \\ &= \int_{\mathbb{R}} h(t) k(t) dt \\ &= (h, k)_{L^2(\mathbb{R})} \end{aligned}$$

### Example 2

Let  $\mathcal{S}_p$  denote the Sobolev space  $H_p(\mathbb{R})$ ,  $p > 0$ . Then  $\mathcal{S} = \mathcal{S}_p \subset L^2(\mathbb{R}) \subset \mathcal{S}_p^* \subset \mathcal{S}^*$  where  $\mathcal{S}$  and  $\mathcal{S}^*$  denote the spaces of all rapidly decreasing functions with the Schwartz topology and the dual of  $\mathcal{S}$ , respectively. When  $p > 1$ ,  $L^2(\mathbb{R}) \subset \mathcal{S}_p^*$  is an abstract Wiener space.

Define  $\langle \Theta(u), \xi \rangle = \xi(\psi(u)) |\psi'(u)|^{1/2}$  where  $\psi(u)$  is a strictly monotonic real valued function with  $\psi(\mathbb{R}) = \mathbb{R}$  such that  $u \rightarrow \Theta(u)$  is continuous and  $\Theta(u) \in \mathcal{S}_p^*$ . This example depends on  $\psi(u)$  and is related to flows (rotation group) in Hida's work [5].

**Corollary 2.3**

If  $\{e_n(t)\}$  is an ONB for  $L^2(\mathbb{R})$ , then  $\{e_n(\psi(u))|\psi'(u)|^{1/2}\}_{n=1}^\infty$  is also an ONB for  $L^2(\mathbb{R})$  where  $\psi(u)$  is given as above definition.

Proof:

$$\begin{aligned}
 & \langle e_n(\psi(u))|\psi'(u)|^{1/2}, e_m(\psi(u))|\psi'(u)|^{1/2} \rangle_{L^2(\mathbb{R})} \\
 &= \int_{\mathbb{R}} e_n(\psi(u))|\psi'(u)|^{1/2} \cdot e_m(\psi(u))|\psi'(u)|^{1/2} du \\
 &= \int_{\mathbb{R}} e_n(\psi(u))e_m(\psi(u))\psi'(u) du \\
 &= \int_{\psi(\mathbb{R})} e_n(t)e_m(t) dt \\
 &= \delta_{m,n}
 \end{aligned}$$

since  $\psi$  is a strictly monotone function.

(Q.E.D.)

Now we check that the condition (ii) is satisfied by using the above Corollary 2.3

$$\begin{aligned}
 & \left( \int_E \Theta(u) du, \int_E \Theta(u) du \right)_H \\
 &= \sum_{n=1}^{\infty} \left( \int_E \Theta(u) du, e_n \right) \left( \int_E \Theta(u) du, e_n \right) \\
 &= \sum_{n=1}^{\infty} \left[ \int_E (\Theta(u), e_n) du \right] \left[ \int_E (\Theta(u), e_n) du \right] \\
 &= \sum_{n=1}^{\infty} \left[ \int_E e_n(\psi(u))|\psi'(u)|^{1/2} du \right]^2 \\
 &= \|1_E(u)\|_{L^2(\mathbb{R})}^2 \\
 &= m(E).
 \end{aligned}$$

For the condition (111)

$$\begin{aligned} & \int_{\mathbb{R}} (\theta(u), h)(\theta(u), k) du \quad \text{for } h, k \in L^2(\mathbb{R}) \\ &= \int_{\mathbb{R}} (\theta(u), \sum_{n=1}^{\infty} (h, e_n) e_n)(\theta(u), \sum_{\ell=1}^{\infty} (k, e_{\ell}) e_{\ell}) du \end{aligned}$$

where  $\{e_n\}_{n=1}^{\infty}$  is an ONB for  $L^2(\mathbb{R})$

$$\begin{aligned} &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} (h, e_n)(\theta(u), e_n) \sum_{\ell=1}^{\infty} (k, e_{\ell})(\theta(u), e_{\ell}) du \\ &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} (h, e_n) e_n(\psi(u)) |\psi'(u)|^{1/2} \sum_{\ell=1}^{\infty} (k, e_{\ell}) e_{\ell}(\psi(u)) |\psi'(u)|^{1/2} du \\ &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} (h, e_n) \xi_n(u) \sum_{\ell=1}^{\infty} (k, e_{\ell}) \xi_{\ell}(u) du \end{aligned}$$

where

$$\{\xi_1(u) = e_1(\psi(u)) |\psi'(u)|^{1/2}\}_{1=1}^{\infty}$$

is another ONB for  $L^2(\mathbb{R})$  by Corollary 2.3

$$\begin{aligned} &= \sum_{n=1}^{\infty} (h, e_n)(k, e_n) \\ &= (h, k)_{L^2(\mathbb{R})}. \end{aligned}$$

## §2. S-transform on $L^2(B, \mu)$

Define  $(S\phi)(\xi) = \int_B \phi(x + \xi) d\mu(x)$  for  $\phi \in L^2(B, \mu)$ ,  $\xi \in B^*$ . Since  $y \in B^* \subset H$

$$\begin{aligned} \int_B \phi(x + \xi) d\mu(x) &= \int_B \phi(x) d\mu(x - \xi) \\ &= \int_B \phi(x) e^{-1/2 \|\xi\|^2 + \langle \xi, x \rangle} d\mu(x) \\ &= e^{-1/2 \|\xi\|^2} \int_B \phi(x) e^{\langle x, \xi \rangle} d\mu(x), \end{aligned}$$

where  $\|\cdot\|$  - H norm and  $\langle x, \xi \rangle$  is dual pairing. (This can be verified by using the translation formula [11]).

We can easily see that  $\phi(x)e^{\langle x, \xi \rangle}$  is also in  $L^2(B)$ . Therefore the S-transform is well-defined. Furthermore we will see that this S-transform carries the random variables on  $B$  to functions defined on  $B^*$ .

We already know that  $K_n$  is identified with the symmetric  $n$ -tensor product  $\bigotimes_n H$ , of  $H$  [12].

Define the functional on  $B^*$  in the following way. For

$$h_1 \otimes \dots \otimes h_n \text{ in } \bigotimes_n H, \text{ define}$$

$$(h_1 \otimes \dots \otimes h_n, \xi) = (h_1, \xi)(h_2, \xi) \dots (h_n, \xi).$$

#### Theorem 2.4

If  $\phi \in K_n$  is represented by

$$h_1 \otimes \dots \otimes h_n \in \bigotimes_n H,$$

then  $S\phi$  is given by

$$(S\phi)(\xi) = (h_1 \otimes \dots \otimes h_n, \xi).$$

Before proving the theorem, we need the following lemma.

#### Lemma 2.1

Suppose  $\phi(x) = \frac{1}{(\sqrt{2})^n} H_n\left(\frac{\langle x, e_\ell \rangle}{\sqrt{2}}\right)$  then

$$(S\phi)(\xi) = \langle \xi, e_\ell \rangle^n = \langle e_\ell \otimes \dots \otimes e_\ell, \xi \rangle, \quad x \in B, \quad \xi \in B^*$$

where  $\{e_\ell\}$  is an ONB for  $H$ .

Proof: The generating function for  $H_n\left(\frac{x}{\sqrt{2}}\right)$  is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(\frac{x}{\sqrt{2}}\right) = e^{-t^2 + \sqrt{2}tx}$$



Let  $\phi(x) = e^{-t^2 + \sqrt{2}t\langle x, e_k \rangle}$ . Then,

$$\phi(x) = e^{-t^2 + \sqrt{2}t\langle x, e_k \rangle} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right)$$

$$\begin{aligned} (S\phi(x))(\xi) &= \int_B \phi(x + \xi) d\mu(x) \\ &= \int_B e^{-t^2 + \sqrt{2}t\langle x+\xi, e_k \rangle} d\mu(x) \\ &= e^{-t^2} e^{\sqrt{2}t\langle \xi, e_k \rangle} \int_B e^{\sqrt{2}t\langle x, e_k \rangle} d\mu(x) \\ &= e^{-t^2} e^{\sqrt{2}t\langle \xi, e_k \rangle} \int_B e^{i\langle x, -i\sqrt{2}te_k \rangle} d\mu(x) \\ &= e^{-t^2} e^{\sqrt{2}t\langle \xi, e_k \rangle} e^{-1/2\| -i\sqrt{2}te_k \|^2_H} \\ &= e^{-t^2} e^{\sqrt{2}t\langle \xi, e_k \rangle} e^{-1/2 \cdot (-1)^2 2t^2} \\ &= e^{\sqrt{2}t\langle \xi, e_k \rangle} = \sum_{n=0}^{\infty} \frac{(\sqrt{2})^n t^n}{n!} \langle \xi, e_k \rangle^n. \end{aligned}$$

On the other hand

$$\begin{aligned} S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right)\right)(\xi) \\ = \int_B \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot H_n\left(\frac{\langle x+\xi, e_k \rangle}{\sqrt{2}}\right) d\mu(x) \end{aligned}$$

(by Lebesgue Dominated Convergence Theorem)

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_B H_n\left(\frac{\langle x+\xi, e_k \rangle}{\sqrt{2}}\right) d\mu(x) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} S(H_n\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right))(\xi). \end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} \frac{(\sqrt{2})^n t^n}{n!} \langle \xi, e_k \rangle^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} S(H_n(\frac{\langle x, e_k \rangle}{\sqrt{2}}))(\xi)$$

implies

$$S(H_n(\frac{\langle x, e_k \rangle}{\sqrt{2}}))(\xi) = (\sqrt{2})^n \langle \xi, e_k \rangle^n$$

i.e.

$$S(\frac{1}{(\sqrt{2})^n} \cdot H_n(\frac{\langle x, e_k \rangle}{\sqrt{2}}))(\xi) = \langle \xi, e_k \rangle^n = \langle e_k \otimes \dots \otimes e_k, \xi \rangle.$$

(Q.E.D.)

#### Proof of Theorem 2.4

It has been shown that

$$\{H_{i_1, \dots, i_r}^{n_1, \dots, n_r}(x) = \frac{1}{\sqrt{2^{n_1} n_2! \dots n_r!}} H_{n_1}(\frac{\langle x, e_{i_1} \rangle}{\sqrt{2}}) \dots H_{n_r}(\frac{\langle x, e_{i_r} \rangle}{\sqrt{2}});$$

$$n_1 + \dots + n_r = n \text{ and } i_1 < \dots < i_r, x \in B, \{e_i\} : \text{ONB for } H\}$$

is an ONB for  $K_n$ , and

$$\{h_{i_1, \dots, i_r}^{n_1, \dots, n_r} = \frac{\sqrt{n!}}{\sqrt{n_1! \dots n_r!}} \underbrace{e_{i_1} \otimes \dots \otimes e_{i_1}}_{n_1} \otimes e_{i_2} \otimes \dots \otimes \underbrace{e_{i_r} \otimes \dots \otimes e_{i_r}}_{n_r};$$

$$n_1 + \dots + n_r = n, i_1 < \dots < i_r\} \text{ is an ONB for } \bigotimes_n H.$$

Therefore it is enough to show that

$$S(H_{i_1, \dots, i_r}^{n_1, \dots, n_r}(x))(\xi) = \frac{1}{\sqrt{n!}} h_{i_1, \dots, i_r}^{n_1, \dots, n_r}(\xi)$$

also, it suffices to show that for

$$\phi(x) = \frac{1}{(\sqrt{2})^{n+m}} H_n(\frac{\langle x, e_k \rangle}{\sqrt{2}}) H_m(\frac{\langle x, e_k \rangle}{\sqrt{2}})$$

$$\begin{aligned}
(S\phi(x))(\xi) &= \langle \xi, e_\ell \rangle^n \langle \xi, e_k \rangle^m \\
&= \langle \underbrace{e_\ell \otimes \dots \otimes e_\ell}_n \otimes \underbrace{e_k \otimes \dots \otimes e_k}_m, \xi \rangle \text{ where } \ell \neq k.
\end{aligned}$$

Consider the generating function

$$e^{-t^2 + \sqrt{2}tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(\frac{x}{\sqrt{2}}\right)$$

and

$$e^{-t^2 + \sqrt{2}tx_1} = \sum_{m=0}^{\infty} \frac{t^m}{m!} H_m\left(\frac{x_1}{\sqrt{2}}\right).$$

Therefore

$$\begin{aligned}
e^{-2t^2 + \sqrt{2}t(x+x_1)} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(\frac{x}{\sqrt{2}}\right) \cdot \sum_{m=0}^{\infty} \frac{t^m}{m!} H_m\left(\frac{x_1}{\sqrt{2}}\right) \\
&= \sum_{n,m=0}^{\infty} \frac{t^n t^m}{n! m!} H_n\left(\frac{x}{\sqrt{2}}\right) H_m\left(\frac{x_1}{\sqrt{2}}\right).
\end{aligned}$$

Now let  $\phi(x) = e^{-2t^2 + \sqrt{2}t(\langle x, e_\ell \rangle + \langle x, e_k \rangle)}$  ( $\ell \neq k$ ) then

$$\begin{aligned}
(S\phi(x))(\xi) &= \int_B \phi(x + \xi) d\mu(x) \\
&= e^{-2t^2} e^{\sqrt{2}t\langle \xi, e_\ell + e_k \rangle} \int_B e^{\sqrt{2}t\langle x, e_\ell + e_k \rangle} d\mu(x) \\
&= e^{-2t^2} e^{\sqrt{2}t\langle \xi, e_\ell + e_k \rangle} e^{-1/2(-2t^2 \|e_\ell + e_k\|_H^2)} \\
&= e^{\sqrt{2}t\langle \xi, e_\ell + e_k \rangle} \\
&= \sum_{j=0}^{\infty} \frac{(\sqrt{2})^j t^j}{j!} \langle \xi, e_\ell + e_k \rangle^j \\
&= \sum_{j=0}^{\infty} \frac{t^j}{j!} (\sqrt{2})^j [\langle \xi, e_\ell \rangle + \langle \xi, e_k \rangle]^j \\
&= \sum_{n,m=0}^{\infty} \frac{t^n t^m}{n! m!} (\sqrt{2})^{n+m} \langle \xi, e_\ell \rangle^n \langle \xi, e_k \rangle^m.
\end{aligned}$$

Therefore

$$\begin{aligned}
 S\left(\frac{1}{(\sqrt{2})^{n+m}} H_n\left(\frac{\langle x, e_\ell \rangle}{\sqrt{2}}\right) H_m\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right)\right)(\xi) &= (e_\ell, \xi)^n (e_k, \xi)^m \\
 &= (\xi, \underbrace{e_\ell \hat{\otimes} \dots \hat{\otimes} e_\ell}_n \hat{\otimes} \underbrace{e_k \hat{\otimes} \dots \hat{\otimes} e_k}_m) \\
 &\quad \quad \quad (Q.E.D.)
 \end{aligned}$$

Moreover, suppose  $(S\phi)(\xi)$  is represented by  $h_1 \hat{\otimes} \dots \hat{\otimes} h_n \in \hat{\otimes}_n H$ , then

$$\|\phi\|_{L^2(B)}^2 = n! \|h_1 \hat{\otimes} \dots \hat{\otimes} h_n\|_{\hat{\otimes}_n H}^2.$$

This is clear since  $S(H_{1, \dots, 1}^{n_1, \dots, n_r}(x))$  is represented by

$$\frac{1}{\sqrt{n!}} h_{1, \dots, 1}^{n_1, \dots, n_r}, \quad \|H_{1, \dots, 1}^{n_1, \dots, n_r}(x)\|_{L^2(B)}^2 \quad \text{and} \quad \|h_{1, \dots, 1}^{n_1, \dots, n_r}\|_{\hat{\otimes}_n H}^2 = 1.$$

Also since the  $S$  transform carries the ONB for  $K_n$  to the ONB for  $\hat{\otimes}_n H$  up to the constant, the  $S$  transform is an isomorphism between  $K_n$  and  $\hat{\otimes}_n H$ . Thus the following theorem has already been shown.

**Theorem 2.5**

$L^2(B, \mu) = e^{\hat{\otimes}_n H}$ , where  $e^{\hat{\otimes}_n H}$  is the direct sum of  $\hat{\otimes}_n H$  with weight  $\sqrt{n!}$ .

**§3. The space  $(L^2)^-$  of generalized random variables.**

In the previous section, it was shown that  $\phi \in K_n$  has the representation  $h_1 \hat{\otimes} \dots \hat{\otimes} h_n \in \hat{\otimes}_n H$  under the  $S$  transform, i.e.

$$(S\phi)(\xi) = (h_1 \hat{\otimes} \dots \hat{\otimes} h_n, \xi) = (h_1, \xi) \dots (h_n, \xi).$$

**Note:** Observe that  $(h_1, \xi) \dots (h_n, \xi)$  is an ordinary random variable. Furthermore,  $\langle f_1, \xi \rangle \dots \langle f_n, \xi \rangle$  is meaningful when  $f_1 \in F$  for  $i = 1, \dots, n$  ( $H \subset F$  is also an abstract Wiener space, and  $H$  is dense in  $F$ ). Therefore we can define natural extension  $U^{-1}$  of  $S^{-1}$  since  $H \subset F$  is dense.

Define

$$(U\phi)(\xi) = (S\phi)(\xi) \quad \text{for } \phi \in L^2(B),$$

and

$$K_n^{(-)} = U^{-1}(\langle \bigotimes_n F, \xi \rangle)$$

$$K_n^{(+)} = U^{-1}(\langle \bigotimes_n F^*, \xi \rangle).$$

Then we have the following diagram

$$\begin{array}{ccccc} K_n^{(+)} & \subset & K_n & \subset & K_n^{(-)} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \sqrt{n!} \bigotimes_n F^* & \subset & \sqrt{n!} \bigotimes_n H & \subset & \sqrt{n!} \bigotimes_n F \end{array} .$$

Define

$$(L^2)^- = \bigcup_{n=0}^{\infty} \bigoplus K_n^{(-)} \quad (L^2)^+ = \bigcup_{n=0}^{\infty} \bigoplus K_n^{(+)}.$$

Then

$$(L^2)^+ \subset L^2(B, \mu) \subset (L^2)^-.$$

We will call  $(L^2)^+$  the space of test random variables  $(L^2)^-$  and the space of generalized random variable.

### Examples of generalized random variables.

#### 1. The random variable $\langle x, \theta(t) \rangle$ .

Note: 1)  $\theta(t) \in F \setminus H$ , therefore  $\langle \cdot, \theta(t) \rangle$  is not an ordinary random variable.

2)  $\theta(t) = \sum_{n=1}^{\infty} (\theta(t), e_n) e_n$  with  $\{e_n\}$  : fixed ONB for  $H$ .

Let's define  $\phi_N(x) = \langle \sum_{i=1}^N (\theta(t), e_i) e_i, x \rangle$  where  $\{e_i\}$  is a fixed ONB for  $H$ . Obviously  $\phi_N(x)$  is an ordinary random variable for  $N = 1, 2, \dots$ , i.e.,  $\phi_N(x) \in L^2(B)$ . Therefore

$$\begin{aligned} (S\phi_N(x))(\xi) &= \int_B \phi_N(x + \xi) d\mu(x) \\ &= \int_B \langle \sum_{i=1}^N (\theta(t), e_i) e_i, x + \xi \rangle d\mu(x) \\ &= \sum_{i=1}^N (\theta(t), e_i) \int_B \langle e_i, x \rangle d\mu(x) \\ &\quad + \sum_{i=1}^N (\theta(t), e_i) \langle e_i, \xi \rangle \\ &= \sum_{i=1}^N (\theta(t), e_i) (\xi, e_i). \end{aligned}$$

Therefore,  $(S\phi_N(x))(\xi) = \langle \sum_{i=1}^N (\theta(t), e_i) e_i, \xi \rangle$ .

Note:  $\sum_{i=1}^N (\theta(t), e_i) e_i$  is not a Cauchy sequence in  $H$ . But we know that  $\sum_{i=1}^N (\theta(t), e_i) e_i$  is a Cauchy sequence in  $F$ ; i.e.,  $(S\phi_N(x))(\xi)$  converges

to  $\langle \sum_{i=1}^{\infty} (\theta(t), e_i) e_i, \xi \rangle = \langle \theta(t), \xi \rangle$  in  $F$ . Therefore we can find the element  $\langle x, \theta(t) \rangle$  in  $K_1^{(-)}$  such that  $U(\langle x, \theta(t) \rangle)(\xi) = \langle \theta(t), \xi \rangle$ . Actually  $\mathbb{E}(\phi_N(x)) = \int_B \phi_N(x) d\mu(x) = 0$  yields that  $\langle x, \theta(t) \rangle = \langle x, \theta(t) \rangle$  where  $\mathbb{E}(\cdot)$  is the expectation of  $(\cdot)$  w.r.t.  $\mu$ .

**Notation:**  $:\phi(x):$  is called the renormalization of  $\phi(x)$ .

### Hida calculus approach for $\langle x, \theta(t) \rangle$

Since  $\theta : u \rightarrow \theta(u)$  is continuous,

$$\begin{aligned}\langle x, \theta(t) \rangle &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left[ \int_0^{t+\Delta} \langle \theta(u), x \rangle du - \int_0^t \langle \theta(u), x \rangle du \right] \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \int_t^{t+\Delta} \langle \theta(u), x \rangle du \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \langle x, \int_t^{t+\Delta} \theta(u) du \rangle.\end{aligned}$$

Let  $\phi_\Delta(x) = \langle x, \frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du \rangle$ . Then  $\phi_\Delta(x) \in L^2(B)$ , since  $\frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du \in H$  and  $\mathbb{E}(\phi_\Delta(x)) = \int_B \langle x, \frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du \rangle d\mu(x) = 0$ . Now

$$(S\phi_\Delta(x))(\xi) = \int_B \langle x + \xi, \frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du \rangle d\mu(x) = \langle \xi, \frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du \rangle$$

so,  $(S\phi_\Delta(x))(\xi)$  is represented by  $\frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du \in H$ . Since  $\|\frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du\|_H^2 = \frac{1}{\Delta}$ ,  $\frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du$  does not converge in  $H$  when  $\Delta \rightarrow 0$ . But  $\frac{1}{\Delta} \int_t^{t+\Delta} \theta(u) du \rightarrow \theta(t)$  as  $\delta \rightarrow 0$  in  $F$ . Therefore let us denote  $U^{-1}(\langle \theta(t), \xi \rangle) = : \langle x, \theta(t) \rangle :$ . Similarly, we can say that  $\langle x, \theta(t) \rangle = : \langle x, \theta(t) \rangle :$

### 2. The Random variable $\langle x, \theta(t) \rangle^2$

Since  $\theta(t) \in F \setminus H$ ,  $\langle x, \theta(t) \rangle^2$  is not an ordinary random variable.

Let  $\phi_N(x)$  be defined by  $\phi_N(x) = \langle \sum_{i=1}^n (\theta(t), e_i) e_i, x \rangle$ . Then  $\phi_N(x) \in L^2(B)$  and

$$\begin{aligned}\mathbb{E} \phi_N(x) &= \int_B \langle \sum_{i=1}^N (\theta(t), e_i) e_i, x \rangle^2 d\mu(x) \\ &= \left\| \sum_{i=1}^N (\theta(t), e_i) e_i \right\|_H^2 \\ &= \sum_{i=1}^N |(\theta(t), e_i)|^2.\end{aligned}$$

**Notation:** We know that  $\sum_{n=1}^{\infty} |(\theta(t), e_n)|^2 = \infty$  since  $\theta(t) \in F \setminus H$  and  $\{e_n\}_{n=1}^{\infty}$  is an O.N.B. for  $H$ . But we can consider  $\sum_{n=1}^{\infty} |(\theta(t), e_n)|^2$  as a number in the nonstandard real number system. So let's denote  $\sum_{n=1}^{\infty} |(\theta(t), e_n)|^2 \equiv \frac{1}{D(t)}$ , where  $\frac{1}{D(t)}$  is a nonstandard real number.

Now

$$\begin{aligned} S(\phi_N(x))(\xi) &= \int_B \langle x + \xi, \sum_{n=1}^N (\theta(t), e_n) e_n \rangle^2 d\mu(x) \\ &= \int_B \langle x, \sum_{n=1}^N (\theta(t), e_n) e_n \rangle^2 d\mu(x) + \int_B \langle \xi, \sum_{n=1}^N (\theta(t), e_n) e_n \rangle^2 d\mu(x) \\ &\quad + 2 \int_B \langle \xi, \sum_{n=1}^N (\theta(t), e_n) e_n \rangle \langle x, \sum_{n=1}^N (\theta(t), e_n) e_n \rangle d\mu(x) \\ &= \left\| \sum_{n=1}^N (\theta(t), e_n) e_n \right\|^2 + \langle \xi, \sum_{n=1}^N (\theta(t), e_n) e_n \rangle^2 \\ &= \sum_{n=1}^N |(\theta(t), e_n)|^2 + \langle \xi, \sum_{n=1}^N (\theta(t), e_n) e_n \rangle^2. \end{aligned}$$

Therefore  $S(\phi_N(x) - \sum_{n=1}^N |(\theta(t), e_n)|^2)(\xi) = \langle \xi, \sum_{n=1}^N (\theta(t), e_n) e_n \rangle^2$ . Hence  $S(\phi_N(x) - \sum_{n=1}^N |(\theta(t), e_n)|^2)(\xi)$  has the representation

$$\sum_{n=1}^N (\theta(t), e_n) \hat{\otimes} \sum_{n=1}^N (\theta(t), e_n) e_n \quad \text{in } H \hat{\otimes} H.$$

Now we know that

$$\left\| \sum_{n=1}^N (\theta(t), e_n) e_n \hat{\otimes} \sum_{n=1}^N (\theta(t), e_n) e_n - \theta(t) \hat{\otimes} \theta(t) \right\|_{F \hat{\otimes} F} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Therefore we can find an element  $\phi(x)$  in  $K_2^{(-)}$  such that  $U(\phi(x))(\xi)$  has the representation  $\theta(t) \hat{\otimes} \theta(t)$  in  $F \hat{\otimes} F$ . Let's denote  $\phi(x)$  as  $:<x, \theta(t)>^2$ : and we call this the renormalization of  $<x, \theta(t)>^2$ . We can easily see that  $\phi(x) \equiv :<x, \theta(t)>^2$ : is the  $(L^2)^-$  limit of



$$\phi_N(x) = \sum_{n=1}^N |(\theta(t), e_n)|^2 \quad \text{i.e.,}$$

$$\begin{aligned} \phi(x) \equiv :<x, \theta(t)>^2: &= (L^2)^{-} = \lim_{N \rightarrow \infty} \left\{ \phi_N(x) - \sum_{n=1}^N |(\theta(t), e_n)|^2 \right\} \\ &= <x, \theta(t)>^2 - \sum_{n=1}^{\infty} |(\theta(t), e_n)|^2 \\ &\equiv <x, \theta(t)>^2 - \frac{1}{D(t)}. \end{aligned}$$

Therefore,  $U(:<x, \theta(t)>^2:)(\xi) = <\xi, \theta(t)>^2$  and  $:<x, \theta(t)>^2: \in K_2^{(-)}$  since  $\theta(t) \hat{\otimes} \theta(t) \in F \hat{\otimes} F$ . From this example motivates why we choose  $K_0^{(-)} = \mathbb{R}^*$ , is the nonstandard real number system.

### 3. The Random variable $:<x, \theta(t)>^n:$

Let  $H_n(x; \sigma^2)$  be Hermite polynomials with parameter i.e.,

$$H_n(x; \sigma^2) = \frac{(-\sigma^2)^n}{n!} e^{x^2/2\sigma^2} \frac{d^n}{dx^n} e^{-x^2/2\sigma^2}, \quad \begin{matrix} \sigma > 0 \\ n > 0 \end{matrix}$$

Then the generating function is

$$\sum_{n=0}^{\infty} \gamma^n H_n(x; \sigma^2) = e^{-\sigma^2 \gamma^2/2 + \gamma x}, \quad \gamma \in \mathbb{C}.$$

Now let  $\phi_N(x) = <x, \sum_{j=1}^N <\theta(t), e_j> e_j>$ .

Let  $\sum_{n=1}^N |<\theta(t), e_j>|^2$  be denoted by  $\rho_N(t)$ . Then  $\rho_N(t) \rightarrow \frac{1}{D(t)}$  as

$N \rightarrow \infty$ , where  $\frac{1}{D(t)} \equiv \sum_{j=1}^{\infty} |(\theta(t), e_j)|^2$ . Then

$$\sum_{n=0}^{\infty} \gamma^n H_n(\phi_N(x); \rho_N(t)) = \exp \left[ -\frac{\rho_N(t) \gamma^2}{2} + \gamma \phi_N(x) \right].$$

Now let us compute the S-transform

$$\begin{aligned}
S\left(e^{\left[-\frac{\rho_N(t)\gamma^2}{2} + \gamma \phi_N(x)\right]}\right)(\xi) &= \int_B e^{-\frac{\rho_N(t)\gamma^2}{2}} e^{\gamma \phi_N(x+\xi)} d\mu(x) \\
&= e^{-\frac{\rho_N(t)\gamma^2}{2}} e^{\gamma \langle \xi, \sum_{j=1}^N \langle O(t), e_j \rangle e_j \rangle} \int_B e^{\gamma \langle x, \sum_{j=1}^N \langle O(t), e_j \rangle e_j \rangle} d\mu(x) \\
&= e^{-\frac{\rho_N(t)\gamma^2}{2}} e^{\gamma \langle \xi, \sum_{j=1}^N \langle O(t), e_j \rangle e_j \rangle} \frac{\gamma^2 \rho_N(t)}{e^{\frac{\gamma^2 \rho_N(t)}{2}}}.
\end{aligned}$$

Therefore

$$S\left(e^{\left[\frac{-\rho_N(t)\gamma^2}{2} + \gamma \phi_N(x)\right]}\right)(\xi) = e^{\gamma \phi_N(\xi)}. \quad (a)$$

Now

$$S\left(\sum_{n=0}^{\infty} \gamma^n H_n(\phi_N(x); \rho_N(t))\right)(\xi)$$

(by Lebesgue Dominated Convergence Th.)

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \gamma^n S(H_n(\phi_N(x); \rho_N(t)))(\xi) \\
&= e^{\gamma \phi_N(\xi)} \quad (\text{by (a)}) \\
&= \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} (\phi_N(\xi))^n.
\end{aligned}$$

Therefore by the uniqueness of the Taylor series,

$$S(H_n(\phi_N(x); \rho_N(t)))(\xi) = \frac{1}{n!} (\phi_N(\xi))^n = \frac{1}{n!} \langle \xi, \sum_{j=1}^N \langle O(t), e_j \rangle e_j \rangle^n.$$

Now note that

$$\sum_{j=1}^N \langle O(t), e_j \rangle e_j \hat{\otimes} \dots \hat{\otimes} \sum_{j=1}^N \langle O(t), e_j \rangle e_j \longrightarrow O(t) \hat{\otimes} \dots \hat{\otimes} O(t)$$

in

$$\underbrace{F \hat{\otimes} \dots \hat{\otimes} F}_n \text{ as } N \rightarrow \infty.$$

It is reasonable to denote the  $(L^2)^-$  limit of  $H_n(\phi_N(x); \rho_N(t))$  by  $H_n(\langle x, \theta(t) \rangle; \frac{1}{D(t)})$ . Let us denote  $n! H_n(\langle x, \theta(t) \rangle; \frac{1}{D(t)}) \equiv : \langle x, \theta(t) \rangle^n :$

Then we can see

$$U(: \langle x, \theta(t) \rangle^n :)(\xi) = \langle \xi, \theta(t) \rangle^n$$

and

$$: \langle x, \theta(t) \rangle^n : \equiv n! H_n(\langle x, \theta(t) \rangle; \frac{1}{D(t)}) \in K_n^{(-)}.$$

#### 4. The random variable $:\exp(\langle x, \theta(t) \rangle):$

Let  $\phi_N(x)$  be defined by

$$\phi_N(x) = e^{\langle x, \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \rangle} \in (L_2).$$

Then

$$\begin{aligned} S(\phi_N(x))(\xi) &= \int_B e^{\langle x+\xi, \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \rangle} d\mu(x) \\ &= e^{\langle \xi, \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \rangle} \int_B e^{\langle x, \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \rangle} d\mu(x) \\ &= e^{\langle \xi, \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \rangle} e^{-\frac{1}{2} \|\sum_{j=1}^N \langle \theta(t), e_j \rangle e_j\|^2} \\ &= e^{\langle \xi, \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \rangle} e^{-\frac{1}{2} \sum_{j=1}^N |\langle \theta(t), e_j \rangle|^2}. \end{aligned}$$

Now we see easily that  $e^{\sum_{j=1}^N \langle \theta(t), e_j \rangle e_j} \longrightarrow e^{\theta(t)}$  as  $N \rightarrow \infty$  in

$\sum_{n=0}^{\infty} \frac{1}{n!} F$ . Therefore

$$e^{\langle \xi, \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \rangle} \longrightarrow e^{\langle \xi, \theta(t) \rangle}$$

in  $(L^2)^-$ . Let us denote the  $(L^2)^-$ -limit of

$$e^{-1/2 \sum_{j=1}^N |\langle \theta(t), e_j \rangle|^2} e^{\phi_N(x)}$$

by  $:e^{\langle x, \theta(t) \rangle}:$ . Then  $U(:e^{\langle x, \theta(t) \rangle}:)(\xi) = e^{\langle \xi, \theta(t) \rangle}$ . It follows that  $:e^{\langle x, \theta(t) \rangle}: = e^{-\frac{1}{2} \frac{1}{D(t)} + \langle x, \theta(t) \rangle}$ .

5. The random variable  $:\exp[c \sum_{n=1}^{\infty} \langle x, e_n \rangle^2]:$  ( $c \neq \frac{1}{2}$ )

**Remarks:** 1) This is related to the Gaussian Brownian functional  $:\exp[c \int_T \dot{B}(t)^2 du]:$  in Hida Calculus.

$$2) e^{c \sum_{j=1}^N \langle x, e_j \rangle^2} = \exp[c \int_{\mathbb{R}} (\sum_{j=1}^N \langle x, e_j \rangle \langle \theta(u), e_j \rangle)^2 du].$$

**Proof of Remark 2):**

$$\begin{aligned} \exp[c \int_{\mathbb{R}} (\sum_{j=1}^N \langle x, e_j \rangle \langle \theta(u), e_j \rangle)^2 du] &= \exp[c \int_{\mathbb{R}} \sum_{j=1}^N \langle x, e_j \rangle^2 \langle \theta(u), e_j \rangle^2 du \\ &\quad + c \int_{\mathbb{R}} (2 \sum_{j \neq l} \langle x, e_j \rangle \langle \theta(u), e_j \rangle \langle x, e_l \rangle \langle \theta(u), e_l \rangle) du] \\ &= \exp[c \sum_{j=1}^N \int_{\mathbb{R}} \langle x, e_j \rangle^2 \langle \theta(u), e_j \rangle^2 du] + \\ &\quad \exp[c \sum_{j \neq l} 2 \int_{\mathbb{R}} \langle x, e_j \rangle \langle x, e_l \rangle \langle \theta(u), e_j \rangle \langle \theta(u), e_l \rangle du] \\ &= \exp[c \sum_{j=1}^N \langle x, e_j \rangle^2]. \end{aligned}$$

**Note:** By Remark 2), we can formally write

$$:\exp[c \sum_{j=1}^{\infty} \langle x, e_j \rangle^2]: \text{ as } :\exp[c \int_{\mathbb{R}} \langle x, \theta(u) \rangle^2 du]:.$$

So we have a formal relation between this random variable and the coordinate system  $\{\langle x, \theta(t) \rangle\}_{t \in \mathbb{R}}$ .

**Terminology:** We will call this random variable  $\exp[c \sum_{j=1}^{\infty} \langle x, e_j \rangle^2]$ : generalized Gaussian random variable.

Now let us find the U transform of  $\exp[c \sum_{j=1}^{\infty} \langle x, e_j \rangle^2]$ :

Let  $\phi_N(x) = \exp[c \sum_{j=1}^N \langle x, e_j \rangle^2]$ . Obviously  $\phi_N(x) \in (L_2)$ .

$$\begin{aligned} (S\phi_N(x))(\xi) &= \int_B e^{c \sum_{j=1}^N \langle x+\xi, e_j \rangle^2} d\mu(x) \\ &= \int_B e^{c \sum_{j=1}^N [\langle x, e_j \rangle^2 + 2\langle x, e_j \rangle \langle \xi, e_j \rangle + \langle \xi, e_j \rangle^2]} d\mu(x) \\ &= e^{c \sum_{j=1}^N \langle \xi, e_j \rangle^2} \int_B e^{c \sum_{j=1}^N [\langle x, e_j \rangle^2 + 2\langle x, e_j \rangle \langle \xi, e_j \rangle]} d\mu(x). \end{aligned}$$

Now let  $\langle x, e_j \rangle = u$ . Then  $\{\langle x, e_j \rangle\}_{j=1}^{\infty}$  is a Gaussian random variable,

$$\begin{aligned} &= e^{c \sum_{j=1}^N \langle \xi, e_j \rangle^2} \prod_{j=1}^N \int_{\mathbb{R}} e^{cu^2 + 2c\langle \xi, e_j \rangle u} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\ &= e^{c \sum_{j=1}^N \langle \xi, e_j \rangle^2} \prod_{j=1}^N \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}[(1-2c)u^2 - 4c\langle \xi, e_j \rangle u]} du \\ &= e^{c \sum_{j=1}^N \langle \xi, e_j \rangle^2} \prod_{j=1}^N \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-1/2 \left[ \left( \sqrt{1-2c} u - \frac{2c\langle \xi, e_j \rangle}{\sqrt{1-2c}} \right)^2 - \frac{4c^2 \langle \xi, e_j \rangle^2}{1-2c} \right]} du \\ &= e^{c \sum_{j=1}^N \langle \xi, e_j \rangle^2} \prod_{j=1}^N \frac{1}{\sqrt{1-2c}} e^{\frac{2c^2 \langle \xi, e_j \rangle^2}{1-2c}} \quad (c \neq \frac{1}{2}) \\ &= \frac{1}{(\sqrt{1-2c})^N} e^{\sum_{j=1}^N \left( c\langle \xi, e_j \rangle^2 + \frac{2c^2}{1-2c} \langle \xi, e_j \rangle^2 \right)} \end{aligned}$$

$$= \frac{1}{(\sqrt{1-2c})^N} e^{\sum_{j=1}^N \left( \frac{c}{1-2c} \langle \xi, e_j \rangle^2 \right)} \quad (c \neq \frac{1}{2}).$$

Therefore

$$S((\sqrt{1-2c})^N e^{c \sum_{j=1}^N \langle x, e_j \rangle^2})(\xi) = e^{\sum_{j=1}^N \left( \frac{c}{1-2c} \langle \xi, e_j \rangle^2 \right)} \quad (c \neq \frac{1}{2}).$$

Now we have

$$e^{\frac{c}{1-2c} \sum_{j=1}^N \langle \xi, e_j \rangle^2} \xrightarrow{N \rightarrow \infty} e^{\frac{c}{1-2c} \sum_{j=1}^{\infty} \langle \xi, e_j \rangle^2}$$

in  $(L^2)^{-}$  where  $c \neq \frac{1}{2}$ . Therefore let us denote  $(L^2)^{-}$ -limit of

$$(\sqrt{1-2c})^N e^{\frac{c}{1-2c} \sum_{j=1}^N \langle x, e_j \rangle^2} \quad (c \neq \frac{1}{2}) \quad \text{as} \quad : e^{c \sum_{j=1}^N \langle x, e_j \rangle^2} :$$

i.e.

$$U(: e^{c \sum_{j=1}^{\infty} \langle x, e_j \rangle^2} :)(\xi) = e^{\frac{c}{1-2c} \sum_{j=1}^{\infty} \langle \xi, e_j \rangle^2} = e^{\frac{c}{1-2c} \|\xi\|^2} \quad (c \neq \frac{1}{2}).$$

Here we have another nonstandard real number  $(\sqrt{1-2c})^{\infty}$ . Let us denote

$$\prod_{j=1}^{\infty} (\sqrt{1-2c}) = (\sqrt{1-2c})^{\infty} \quad \text{as} \quad v(c)$$

in the nonstandard real number system. Then we have the relation

$$: e^{c \sum_{j=1}^{\infty} \langle x, e_j \rangle^2} : = v(c) e^{c \sum_{j=1}^{\infty} \langle x, e_j \rangle^2}.$$

## CHAPTER 3. Operators acting on $(L^2)^{-}$

### §1. $\Theta(t)$ -differentiation and its adjoint operator.

It is time to discuss differentiation not only for ordinary random variables but also for generalized random variables. It will be defined through U-transform.

#### Definition 3.1

Suppose  $(U\phi)'(\xi)$  has first functional derivative  $(U\phi)'(\xi)$  at  $\xi$ , with  $(U\phi)'(\xi) \in F^*$ , i.e.,

$|U(\xi + \eta) - U(\xi) - ((U\phi)'(\xi), \eta)| = o(\|\eta\|_F)$  and  $(U\phi)'(\xi)$  is a U-transform of a generalized random variable. We define  $\partial_{\Theta(t)}\phi$  to be the generalized random variable with U-transform

$$U(\partial_{\Theta(t)}\phi)(\xi) = \langle (U\phi)'(\xi), \Theta(t) \rangle.$$

**Observation:** Since  $(U\phi)'(\xi)$  is a continuous linear functional,  $(U\phi)'(\xi) \in F^*$  and  $\langle (U\phi)'(\xi), \Theta(t) \rangle$  makes sense.

#### Theorem 3.1

Suppose  $\phi(x) \in (L^2)$  has F-Fréchet derivative, i.e.,

$$|\phi(x + y) - \phi(x) - (\phi'(x), y)| = o(\|y\|_F), \quad y \in F.$$

Then  $\partial_{\Theta(t)}\phi(x) = \langle \phi'(x), \Theta(t) \rangle$

**Proof:** We need to show that

$$U(\langle \phi'(x), \theta(t) \rangle)(\xi) = \langle (U\phi')(\xi), \theta(t) \rangle$$

thus

$$U(\langle \phi'(x), \theta(t) \rangle)(\xi) = \int_B \langle \phi'(x + \xi), \theta(t) \rangle d\mu(x)$$

(by definition of  $U$  transform)

$$= \langle \int_B \phi'(x + \xi) d\mu(x), \theta(t) \rangle.$$

$$= \langle \frac{\delta}{\delta \xi} \left[ \int_B \phi(x + \xi) d\mu(x) \right], \theta(t) \rangle$$

$$= \langle (U\phi')(\xi), \theta(t) \rangle. \quad (\text{Q.E.D.})$$

**Remarks:** 1) If  $U\phi'(\xi)$  exists, then

$$\langle U\phi'(\xi), \eta \rangle = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \{U(\xi + \lambda\eta) - U(\xi)\} \quad \text{where } \xi, \eta \in B^*$$

(variation in the Gateaux sense).

2) In Hida calculus,  $U(\partial_{\theta(t)} \phi(x))(\xi) = U'_\xi(\xi, t)$ , the functional derivative of  $U$ .

**Example 1.**  $\phi(x) = \langle x, \xi_0 \rangle$  with  $\xi_0 \in B^*$ . Then  $(U\phi)(\xi) = \langle \xi, \xi_0 \rangle$ , so  
 $\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \{U\phi(\xi + \lambda\eta) - U\phi(\xi)\} = \langle \eta, \xi_0 \rangle$ . Therefore,  $\langle (U\phi)'(\xi), \eta \rangle = \langle \xi_0, \eta \rangle$   
 and thus  $U(\langle \partial_{\theta(t)} \phi(x) \rangle)(\xi) = \langle \xi_0, \theta(t) \rangle$ .

**Example 2.** Suppose  $\phi(x) = : \langle x, \xi_1 \rangle \langle x, \xi_2 \rangle :$  where  $\xi_1, \xi_2 \in B^*$ .

Then  $(U\phi)(\xi) = \langle \xi, \xi_1 \rangle \langle \xi, \xi_2 \rangle$ , and

$$(U\phi)'(\xi; \cdot) = \langle \xi_1, \cdot \rangle \langle \xi_2, \xi \rangle + \langle \xi_2, \cdot \rangle \langle \xi_1, \xi \rangle$$

i.e.,



$$\begin{aligned}
\langle (U\phi(x))'(\xi), \theta(t) \rangle &= U(\partial_{\theta(t)}\phi)(\xi) \\
&= \langle \xi_1, \theta(t) \rangle \langle \xi_2, \xi \rangle + \langle \xi_2, \theta(t) \rangle \langle \xi_1, \xi \rangle.
\end{aligned}$$

**Remark:** Suppose  $\phi(x) = \langle x, \theta(u) \rangle$ . Then  $(U\phi)(\xi) = \langle \xi, \theta(u) \rangle$ . Therefore,  $\langle (U\phi)'(\xi), \eta \rangle = \langle \eta, \theta(u) \rangle$ , and  $U(\partial_{\theta(t)}\phi)(\xi) = \langle \theta(t), \theta(u) \rangle$ .

Similarly if  $\phi(x) = \langle x, \theta(t) \rangle$ , then

$$U(\partial_{\theta(t)}\phi)(\xi) = \langle \theta(t), \theta(t) \rangle \equiv \frac{1}{D(t)} \text{ is a nonstandard real number, i.e.,}$$

$$\begin{aligned}
\partial_{\theta(t)}(\langle x, \theta(u) \rangle) &= \langle \theta(t), \theta(u) \rangle & \text{when } t \neq u \\
&= \frac{1}{D(t)} & \text{when } t = u.
\end{aligned}$$

This is the infinite dimensional analog of  $\partial_{ij} x_{ij} = \delta_{ij}$  in the finite dimensional case. In the Hida calculus, i.e.  $\theta(u) = \delta_u^{(\cdot)}$

$$\begin{aligned}
\partial_{\dot{B}(t)} \langle \dot{B}(u) \rangle &= \delta_t(u) & \text{if } t \neq u \\
&= \delta_t(t) \frac{1}{dt} & \text{if } t = u.
\end{aligned}$$

**Example 3.** Let  $\phi(x) = H_n\left(\frac{\langle x, e_1 \rangle}{\sqrt{2}}\right)$  with  $e_1 \in B^*$ ,  $H_n$  be the Hermite polynomial as before. Then  $(U\phi)(\xi) = (\sqrt{2})^n (\xi, e_1)^n$ . Then

$$U(\partial_{\theta(t)}\phi)(\xi) = (\sqrt{2})^n (\xi, e_1)^{n-1} \langle e_1, \theta(t) \rangle,$$

and we have

$$U(\partial_{\theta(t)}\phi)(x) = n \langle e_1, \theta(t) \rangle H_{n-1}\left(\frac{\langle x, e_1 \rangle}{\sqrt{2}}\right).$$

**Theorem 3.2.** If  $\phi(x) \in K_n$ , then  $\partial_{\theta(t)}\phi \in K_{n-1}$ .

**Proof:** Suppose  $\psi$  is a random variable of degree less than or equal to  $n-2$ , then

$$\int_B \psi(x) \partial_{\Theta(t)} \phi(x) d\mu(x) = \int_B \psi(x) \langle \phi'(x), \Theta(t) \rangle d\mu(x)$$

(by integration by parts formula [10])

$$= \int_B \phi(x) \{ \psi(x) \langle \Theta(t), x \rangle - \langle \Theta(t), \psi'(x) \rangle \} d\mu(x)$$

$$= 0$$

since  $\phi \in K_n$  and the quantity inside  $\{ \}$  is a random variable whose degree is less than or equal to  $n - 1$ . Therefore  $\partial_{\Theta(t)} \phi$  is orthogonal to  $K_0, \dots, K_{n-2}$ . Similar computation as Example 3 leads us to conclude  $\partial_{\Theta(t)} \phi \in K_{n-1}$ . (Q.E.D.)

**Definition 3.2.** Define the adjoint  $\partial_{\Theta(t)}^*$  of  $\partial_{\Theta(t)}$  by

$$\langle \partial_{\Theta(t)}^* \phi, \psi \rangle = \langle \phi, \partial_{\Theta(t)} \psi \rangle \quad \text{where} \quad \phi \in (L^2)^-, \psi \in (L^2)^+.$$

**Corollary 3.1**

$$U(\partial_{\Theta(t)}^* \phi)(\xi) = \langle \xi, \Theta(t) \rangle (U\phi)(\xi).$$

**Proof:**

$$\begin{aligned} U(\partial_{\Theta(t)}^* \phi)(\xi) &= \int_B \partial_{\Theta(t)}^* \phi(x + \xi) d\mu(x) \\ &= \int_B \partial_{\Theta(t)}^* \phi(y) d\mu(y - \xi) \end{aligned}$$

(by the translation formula [11])

$$\begin{aligned} &= \int_B \partial_{\Theta(t)}^* \phi(y) e^{-\frac{1}{2} \|\xi\|^2 + \langle y, \xi \rangle} d\mu(y) \\ &= \int_B \phi(y) \partial_{\Theta(t)} \left[ e^{-\frac{1}{2} \|\xi\|^2 + \langle y, \xi \rangle} \right] d\mu(y) \\ &= \int_B \phi(y) \cdot e^{-\frac{1}{2} \|\xi\|^2 + \langle y, \xi \rangle} \{ \langle \xi, \Theta(t) \rangle \} d\mu(y) \end{aligned}$$

$$\begin{aligned}
&= \langle \xi, \theta(t) \rangle \int_B \phi(y) e^{-\frac{1}{2} \|\xi\|^2 + \langle y, \xi \rangle} d\mu(y) \\
&= \langle \xi, \theta(t) \rangle \int_B \phi(y) d\mu(y - \xi) \\
&= \langle \xi, \theta(t) \rangle \int_B \phi(x + \xi) d\mu(x) \\
&= \langle \xi, \theta(t) \rangle (U\phi)(\xi).
\end{aligned}$$

(Q.E.D.)

**Definition 3.3** (Multiplication by  $\langle \cdot, \theta(t) \rangle$ )

Suppose  $\phi \in (L^2)^-$ . Define,

$$\langle \cdot, \theta(t) \rangle \cdot \phi = \partial_{\theta(t)} \phi + \partial_{\theta(t)}^* \phi.$$

**Observation:** Let  $\phi \in (L^2)^-$ ,  $\psi \in (L^2)^+$ .

$$\langle \partial_{\theta(t)}^* \phi, \psi \rangle = \langle \phi, \partial_{\theta(t)} \psi \rangle,$$

i.e.,

$$\begin{aligned}
\int_B (\partial_{\theta(t)}^* \phi)(x) \psi(x) d\mu(x) &= \int_B \phi(x) (\partial_{\theta(t)} \psi)(x) d\mu(x) \\
&= \int_B \phi(x) \langle \psi'(x), \theta(t) \rangle d\mu(x) \quad (\text{by Theorem 3.1}) \\
&= \int_B [\phi(x) \psi(x) \langle x, \theta(t) \rangle - \psi(x) \langle \phi'(x), \theta(t) \rangle] d\mu(x) \\
&= \int_B \psi(x) [- (\partial_{\theta(t)} \phi)(x) + \langle x, \theta(t) \rangle \phi(x)] d\mu(x).
\end{aligned}$$

Therefore  $(\partial_{\theta(t)}^* \phi)(x) = -(\partial_{\theta(t)} \phi)(x) + \langle x, \theta(t) \rangle \phi(x)$  i.e.,

$$\partial_{\theta(t)}^* \phi + \partial_{\theta(t)} \phi = \langle \cdot, \theta(t) \rangle \phi.$$

**Theorem 3.3.**

(1) If  $\phi(x) \in K_n^{(+)}$  is represented by

$$k_1^* \otimes \dots \otimes k_n^* \in \bigotimes_n F^* \quad \text{then } \partial_{\theta(t)} \phi \text{ is represented by}$$

$$\sum_{j=1}^n (k_j^*, 0(t)) k_1^* \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_n^*,$$

where

$$k_1^* \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_n^* = k_1^* \underset{(j)}{\otimes} k_2^* \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_{j-1}^* \underset{(j)}{\otimes} k_{j+1}^* \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_n^*$$

(2) If  $\phi(x) \in K_n^{(-)}$  is represented by  $k_1 \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_n \in \underset{n}{\otimes} F$  then  $\partial_{0(t)}^* \phi$  is represented by  $k_1 \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_n \underset{(j)}{\otimes} 0(t)$ .

Proof: (1) Since  $\phi \in K_n^{(+)}$  is represented by  $k_1^* \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_n^*$ ,

$$(U\phi)(\xi) = (k_1^*, \xi)(k_2^*, \xi) \dots (k_n^*, \xi).$$

Then

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \{ (U\phi)(\xi + \lambda \eta) - (U\phi)(\xi) \} \\ &= (k_1^*, \eta)(k_2^*, \xi) \dots (k_n^*, \xi) + \dots + (k_1^*, \xi) \dots (k_{n-1}^*, \xi)(k_n^*, \eta) \end{aligned}$$

Therefore

$$\langle (U\phi)'(\xi), 0(t) \rangle = \sum_{j=1}^n \langle k_j^*, 0(t) \rangle \langle k_1^*, \xi \rangle \dots \langle k_{j-1}^*, \xi \rangle \langle k_{j+1}^*, \xi \rangle \dots \langle k_n^*, \xi \rangle.$$

Therefore  $\partial_{0(t)}^* \phi$  is represented by  $\sum_{j=1}^n (k_j^*, 0(t)) k_1^* \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_n^*$ .

(2) Since  $\phi(x) \in K_n^{(-)}$  is represented by  $k_1 \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_n$ ,

$$(U\phi)(\xi) = (k_1, \xi)(k_2, \xi) \dots (k_n, \xi)$$

where  $k_i \in F$  for  $i = 1, \dots, n$ . Now let  $\psi \in K_{n+1}^{(+)}$  be represented by  $k_1^* \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_{n+1}^*$  where  $k_i^* \in F^*$  for  $i = 1, \dots, n+1$ . Then by (1)

$\partial_{0(t)}^* \psi$  is represented by

$$\sum_{j=1}^{n+1} \langle k_j^*, 0(t) \rangle \langle k_1^* \underset{(j)}{\otimes} \dots \underset{(j)}{\otimes} k_{n+1}^* \rangle.$$

Now

$$\begin{aligned}
\langle \partial_{\Theta(t)}^* \phi, \psi \rangle &= \langle \phi, \partial_{\Theta(t)} \psi \rangle \\
&= n! \langle k_1 \hat{\otimes} \dots \hat{\otimes} k_n, \sum_{j=1}^{n+1} \langle k_j^*, \Theta(t) \rangle (k_1^* \hat{\otimes} \dots \hat{\otimes} k_{n+1}^*)_{(j)} \rangle \\
&= n! \sum_{j=1}^{n+1} \langle k_j^*, \Theta(t) \rangle \langle k_1 \hat{\otimes} \dots \hat{\otimes} k_n, k_1^* \hat{\otimes} \dots \hat{\otimes} k_{n+1}^* \rangle \\
&= n! \sum_{j=1}^n \langle k_j^*, \Theta(t) \rangle \langle \frac{1}{n!} \sum_{\sigma} k_{\sigma(1)} \otimes \dots \otimes k_{\sigma(n)}, k_1^* \otimes \dots \otimes k_{n+1}^* \rangle
\end{aligned}$$

where the second summation  $\sigma$  is over all permutations of  $(1, 2, \dots, n)$

$$\begin{aligned}
&= \sum_{j=1}^n \langle k_j^*, \Theta(t) \rangle \left( \sum_{\sigma} k_{\sigma(1)} \otimes \dots \otimes k_{\sigma(n)}, k_1^* \otimes \dots \otimes k_{n+1}^* \right) \\
&= \left\langle \sum_{\sigma} k_{\sigma(1)} \otimes \dots \otimes k_{\sigma(j-1)} \otimes \Theta(t) \otimes k_{\sigma(j+1)} \otimes \dots \otimes k_{\sigma(n)}, k_1^* \otimes \dots \otimes k_{n+1}^* \right\rangle \\
&= (n+1)! \langle k_1 \hat{\otimes} \dots \hat{\otimes} k_n \hat{\otimes} \Theta(t), k_1^* \hat{\otimes} \dots \hat{\otimes} k_{n+1}^* \rangle.
\end{aligned}$$

Therefore,  $\partial_{\Theta(t)}^* \phi$  is represented by  $k_1 \hat{\otimes} \dots \hat{\otimes} k_n \hat{\otimes} \Theta(t)$ .

### Theorem 3.4

- 1)  $[\partial_{\Theta(t)}, \partial_{\Theta(s)}] = [\partial_{\Theta(s)}^*, \partial_{\Theta(t)}^*] = 0$
- 2)  $[\partial_{\Theta(t)}, \partial_{\Theta(s)}^*] = \langle \Theta(s), \Theta(t) \rangle I$

where  $[a, b] = ab - ba$  and  $I$  is the identity.

Proof: 1) Let  $\phi(x) \in K_n^{(+)}$  and  $k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^*$  be the representation for  $\phi(x)$  where  $k_i^* \in F^*$  for  $i = 1, \dots, n$ . Then

$(U\phi)(\xi) = (k_1^*, \xi) \dots (k_n^*, \xi)$ . By Theorem 5.3

$$U(\partial_{\Theta(t)}^* \phi)(\xi) = \sum_{j=1}^n \langle k_j^*, \Theta(t) \rangle (k_1^*, \xi) \dots (k_{j-1}^*, \xi) (k_{j+1}^*, \xi) \dots (k_n^*, \xi).$$

Now

$$\begin{aligned}
& \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \{ U(\partial_{\theta(t)} \phi)(\xi + \lambda \eta) - U(\partial_{\theta(t)} \phi)(\xi) \} \\
&= (k_1, \theta(t)) [ (k_2, \eta)(k_3, \xi) \dots (k_n, \xi) + \dots + (k_2, \xi) \dots (k_{n-1}, \xi)(k_n, \eta) ] \\
&+ \dots + (k_n, \theta(t)) [ (k_1, \eta)(k_2, \xi) \dots (k_{n-1}, \xi) \\
&+ \dots + (k_1, \xi) \dots (k_{n-2}, \xi)(k_{n-1}, \eta) ].
\end{aligned}$$

Therefore,

$$U(\partial_{\theta(s)} \partial_{\theta(t)} \phi)(\xi) = \sum_{i=1}^n \langle k_i, \theta(t) \rangle \sum_{\substack{j=1 \\ j \neq i}}^n \langle k_j, \theta(s) \rangle \langle k_1, \xi \rangle \dots \langle k_n, \xi \rangle_{(i,j)}$$

where

$$\begin{aligned}
& \langle k_1, \xi \rangle \dots \langle k_n, \xi \rangle_{(i,j)} \equiv \langle k_1, \xi \rangle \dots \langle k_{i-1}, \xi \rangle \\
& \quad \cdot \langle k_{i+1}, \xi \rangle \dots \langle k_{j-1}, \xi \rangle \langle k_{j+1}, \xi \rangle \dots \langle k_n, \xi \rangle.
\end{aligned}$$

Similarly

$$U(\partial_{\theta(t)} \partial_{\theta(s)} \phi)(\xi) = \sum_{i=1}^n \langle k_i, \theta(s) \rangle \sum_{\substack{j=1 \\ j \neq i}}^n \langle k_j, \theta(t) \rangle \langle k_1, \xi \rangle \dots \langle k_n, \xi \rangle_{(i,j)}.$$

Therefore

$$[\partial_{\theta(t)}, \partial_{\theta(s)}] = (\partial_{\theta(t)} \partial_{\theta(s)} - \partial_{\theta(s)} \partial_{\theta(t)})(\phi) = 0,$$

since

$$U(\partial_{\theta(t)} \partial_{\theta(s)} \phi)(\xi) = U(\partial_{\theta(s)} \partial_{\theta(t)} \phi)(\xi).$$

Now, let's show that  $[\partial_{\theta(s)}^*, \partial_{\theta(t)}^*] = 0$ . It is enough to show this for  $\phi \in K_n^{(-)}$ . So let  $\phi$  be represented by  $k_1 \otimes \dots \otimes k_n \otimes \theta(t)$  and  $\partial_{\theta(s)}^* \partial_{\theta(t)}^* \phi$  be represented by  $k_1 \otimes \dots \otimes k_n \otimes \theta(t) \otimes \theta(s)$ . Similarly  $\partial_{\theta(t)}^* \partial_{\theta(s)}^* \phi$  is represented by  $k_1 \otimes \dots \otimes k_n \otimes \theta(s) \otimes \theta(t)$ . Therefore

$$[\partial_{\theta(s)}^*, \partial_{\theta(t)}^*] = \partial_{\theta(s)}^* \partial_{\theta(t)}^* - \partial_{\theta(t)}^* \partial_{\theta(s)}^* = 0.$$

(2) Let  $\phi \in K_n^{(+)}$  be represented by  $k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^* \hat{\otimes} \theta(s)$ , i.e.,

$$U(\partial_{\theta(s)}^* \phi)(\xi) = (k_1^*, \xi) \dots (k_n^*, \xi)(\theta(s), \xi)$$

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \{ U(\partial_{\theta(s)}^* \phi)(\xi + \lambda \eta) - U(\partial_{\theta(s)}^* \phi)(\xi) \} \\ = (k_1^*, \eta)(k_2^*, \xi) \dots (\theta(s), \xi) + \dots + (k_n^*, \eta)(k_1^*, \xi) \dots (k_{n-1}^*, \xi)(\theta(s), \xi) \\ + (k_1^*, \xi) \dots (k_n^*, \xi)(\theta(s), \eta). \end{aligned}$$

Therefore,

$$\begin{aligned} U(\partial_{\theta(t)} \partial_{\theta(s)}^* \phi)(\xi) &= (k_1^*, \theta(t))(k_2^*, \xi) \dots (k_n^*, \xi)(\theta(s), \xi) \\ &+ \dots + (k_n^*, \theta(t))(k_1^*, \xi) \dots (k_{n-1}^*, \xi)(\theta(s), \xi) \\ &+ (k_1^*, \xi) \dots (k_n^*, \xi)(\theta(s), \theta(t)). \end{aligned}$$

Now  $\partial_{\theta(t)} \phi$  is represented by

$$\sum_{j=1}^n \langle k_j^*, \theta(t) \rangle k_1^* \hat{\otimes} \dots \hat{\otimes}_{(j)} k_n^*.$$

Therefore  $\partial_{\theta(s)}^* \partial_{\theta(t)} \phi$  is represented by

$$\sum_{j=1}^n \langle k_j^*, \theta(t) \rangle k_1^* \hat{\otimes} \dots \hat{\otimes} \theta(s),$$

i.e.,

$$\begin{aligned} U(\partial_{\theta(s)}^* \partial_{\theta(t)} \phi)(\xi) &= (k_1^*, \theta(t))(k_2^*, \xi) \dots (\theta(s), \xi) \\ &+ \dots + (k_n^*, \theta(t))(k_1^*, \xi) \dots (\theta(s), \xi). \end{aligned}$$

Therefore

$$U(\partial_{\theta(t)} \partial_{\theta(s)}^* \phi)(\xi) - U(\partial_{\theta(s)}^* \partial_{\theta(t)} \phi)(\xi) = (k_1^*, \xi) \dots (k_n^*, \xi)(\theta(s), \theta(t)).$$

Thus

$$[\partial_{\Theta(t)}, \partial_{\Theta(s)}^*] = \partial_{\Theta(t)} \partial_{\Theta(s)}^* - (\Theta(s), \Theta(t))I. \quad (\text{Q.E.D.})$$

## §2. Laplacian operators.

In the theory of abstract Wiener space, the Gross Laplacian  $\Delta_G$  [11] and the Beltrami Laplacian  $\Delta_B$  [16] are defined as follows;

$$(\Delta_G f)(x) = \text{trace}_H f''(x)$$

$$(\Delta_B g)(x) = \text{trace}_H g''(x) - \langle x, g'(x) \rangle$$

where the primes denote the  $H$ -Fréchet derivatives.  $f''(x)$  is assumed to be a trace class operator of  $H$ .  $|g'(\cdot)|_H \in L^2(B)$  and  $|g''(\cdot)|_{H-S} \in L^2(B)$  are assumed for  $g$ .

We denote  $\phi'(x)$ ,  $\phi''(x)$  as  $F$ -Fréchet derivative of  $\phi(x)$  in the previous section (Theorem 3.1.1) and the  $\Theta(t)$ -differentiation is related with  $F$ -directional derivatives. On the other hand, if  $\phi'(x)$  is the  $F$ -directional derivative of  $\phi(x)$ , then  $\phi'(x) \in F^*$  and the restriction of  $\phi'(x)$  to  $H$  is the derivative of  $\phi(x)$  in  $H$ -direction  $\phi(x)$ .

Since  $f''(x)$  is assumed to be a trace class operator of  $H$ ,  $\Delta_G f$  is defined pointwise. On the other hand, it has been shown in [16] that  $\text{trace}_H g''(x) - \langle x, g'(x) \rangle$  is the  $L^2(B)$ -limit of  $\text{trace}_H P g''(x) - (Px, g'(x))$  as  $P$  converges strongly to the identity in the set of finite dimensional orthogonal projections of  $H$ .

### Theorem 3.5

Suppose  $\phi$  is a twice differentiable function in  $F$ -direction on  $B$  with  $\phi'(x) \in F^*$  and  $\phi''(x)$  a trace class operator of  $H$  for all  $x \in B$ , then  $\phi$  is twice  $\Theta(t)$ -differentiable and



$$\Delta_G \phi = \int_{\mathbb{R}} \partial_{\Theta(t)}^2 \phi dt, \quad \Delta_B \phi = - \int_{\mathbb{R}} \partial_{\Theta(t)}^* \cdot \partial_{\Theta(t)} \phi dt,$$

whenever the integral exists.

**Proof:** We have  $\partial_{\Theta(t)} \phi(x) = \langle \phi'(x), \Theta(t) \rangle$  where  $\phi'(x) \in F^*$ . Then

$$\partial_{\Theta(t)} \phi(x) = \sum_{n=1}^{\infty} \langle \phi'(x), \xi_n \rangle \langle \Theta(t), \xi_n \rangle_F, \quad \text{where } \{\xi_n\}_{n=1}^{\infty} \text{ is an ONB for } F.$$

Therefore

$$\begin{aligned} \partial_{\Theta(t)}^2 \phi(x) &= \sum_{k,n=1}^{\infty} \langle \phi''(x) \xi_n, \xi_k \rangle \langle \Theta(t), \xi_n \rangle_F \langle \Theta(t), \xi_k \rangle_F \\ &= \sum_{k,n=1}^{\infty} \langle \phi''(x) \frac{1}{\lambda_n} e_n, \frac{1}{\lambda_k} e_k \rangle \langle \Theta(t), \frac{1}{\lambda_n} e_n \rangle_F \langle \Theta(t), \frac{1}{\lambda_k} e_k \rangle_F \\ &= \sum_{k,n=1}^{\infty} (\phi''(x) e_n, e_k) \langle \Theta(t), e_n \rangle \langle \Theta(t), e_k \rangle. \end{aligned}$$

Then it follows that

$$\int_{\mathbb{R}} \partial_{\Theta(t)}^2 \phi dt = \sum_{n=1}^{\infty} \langle \phi''(x) e_n, e_n \rangle (\text{since } \int_{\mathbb{R}} \partial_{\Theta(t)}^2 \phi(x) dt < \infty) = (\Delta_G \phi)(x).$$

We give a formal proof for the second part.

$$(\Delta_B \phi)(x) = (\Delta_G \phi)(x) - \langle x, \phi'(x) \rangle$$

(Let us write  $\langle x, \phi'(x) \rangle$  as  $\int_{\mathbb{R}} \langle x, \Theta(t) \rangle \langle \phi'(x), \Theta(t) \rangle dt$  formally)

$$\begin{aligned} &= \Delta_G \phi(x) - \int_{\mathbb{R}} \langle x, \Theta(t) \rangle \langle \phi'(x), \Theta(t) \rangle dt \\ &= \int_{\mathbb{R}} (\partial_{\Theta(t)}^2 \phi) dt - \int_{\mathbb{R}} \langle \partial_{\Theta(t)}^* + \partial_{\Theta(t)} \rangle \langle \phi'(x), \Theta(t) \rangle dt \\ &= \int_{\mathbb{R}} (\partial_{\Theta(t)}^2 \phi) dt - \int_{\mathbb{R}} (\partial_{\Theta(t)}^* + \partial_{\Theta(t)}) (\partial_{\Theta(t)} \phi) dt \\ &= - \int_{\mathbb{R}} (\partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi) dt. \end{aligned} \quad (\text{Q.E.D.})$$

**Corollary 3.2**

Suppose  $\phi(x) \in K_n$  is represented by  $h_1 \hat{\otimes} \dots \hat{\otimes} h_n$ ,  $h_i \in H$ ,  $i = 1, 2, \dots, n$  and  $\int_{\mathbb{R}} \partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi(x) dt$  exists, then  $\Delta_B \phi(x) = -n\phi(x)$ .

**Proof:** By Theorem 3.5 we know that

$$\Delta_B \phi(x) = -\int_{\mathbb{R}} \partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi(x) dt.$$

So it is enough to show that

$$U(\Delta_B \phi(x))(\xi) = U(-\int_{\mathbb{R}} \partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi(x) dt)(\xi) = -n U(\phi(x))(\xi).$$

Now since  $\phi(x)$  is represented by  $h_1 \hat{\otimes} \dots \hat{\otimes} h_n$ ,

$$U(\phi(x))(\xi) = (h_1, \xi) \dots (h_n, \xi).$$

Then by Theorem 3.3

$$U(\partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi(x))(\xi) = \sum_{j=1}^n \langle \Theta(t), \xi \rangle \langle h_1, \xi \rangle \dots \langle h_n, \xi \rangle \langle h_j, \Theta(t) \rangle.$$

Therefore

$$\begin{aligned} U(\int_{\mathbb{R}} \partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi(x) dt)(\xi) &= \int_{\mathbb{R}} U(\partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi(x))(\xi) dt \\ &= \sum_{j=1}^n \langle h_1, \xi \rangle \dots \langle h_n, \xi \rangle \langle h_j, \xi \rangle \\ &= nU(\phi(x))(\xi). \end{aligned} \quad (\text{Q.E.D.})$$

**Remark:** By Theorem 3.5, we can generalize the definition of Beltrami Laplacian  $\Delta_B \phi \equiv -\int_{\mathbb{R}} \partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi dt$  for  $\phi \in (L^2)^{-}$  as long as  $\int_{\mathbb{R}} \partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi dt$  exists. Then for  $\phi$  in  $K_n^{(-)}$

$$\Delta_B \phi = -n\phi.$$

For the space  $(L^2)^{-}$  of generalized random variables, we can define another two Laplacians as in Hida calculus [15].

**Definition 3.4**

Suppose  $U(\partial_{\theta(t)}^2 \phi)(\xi)$  is given by

$$U(\partial_{\theta(t)}^2 \phi)(\xi) = U_1''(\xi; \theta(t)) \langle \theta(t), \theta(t) \rangle + U_2''(\xi; \theta(t)) \langle \theta(t), \xi \rangle \langle \theta(t), \xi \rangle$$

and  $U_1''(\xi; \theta(t))$  and  $U_2''(\xi; \theta(t))$  are U-transforms of generalized random variables, then we define the Levy Laplacian  $\Delta_L$  and the Volterra Laplacian  $\Delta_V$  as follows:

$$U(\Delta_L \phi)(\xi) = \int_{\mathbb{R}} U_1''(\xi; \theta(t)) dt$$

$$U(\Delta_V \phi)(\xi) = \int_{\mathbb{R}} U_2''(\xi; \theta(t)) \langle \theta(t), \xi \rangle \langle \theta(t), \xi \rangle dt.$$

**Remark:** From the non-standard analysis point of view the above definition can be interpreted as follows:

$$U(\Delta_L \phi)(\xi) = \int_{\mathbb{R}} U(\partial_{\theta(t)}^2 \phi)(\xi) (D(t) dt).$$

(This is the analog of  $\Delta_L \phi = \int_{\mathbb{R}} \partial_t^2 \phi(dt)^2$  in Hida Calculus [15]).

**Definition 3.5**

Suppose  $\phi$  is a generalized random variable in  $(L^2)^{-}$  and satisfies

$$U(\phi)(\xi) = \int_{\mathbb{R}^k} f(u_1, \dots, u_k) \langle \xi, \theta(u_1) \rangle^{n_1} \dots \langle \xi, \theta(u_k) \rangle^{n_k} du_1 \dots du_k$$

where  $\sum_{j=1}^k n_j = n$ , and  $f(u_1, \dots, u_k)$  is such that

$$\int_{\mathbb{R}^k} f(u_1, \dots, u_k) \underbrace{\theta(u_1) \otimes \dots \otimes \theta(u_1)}_{n_1 \text{ times}} \otimes \underbrace{\theta(u_2) \otimes \dots \otimes \theta(u_k)}_{n_k \text{ times}} \otimes \dots \otimes \theta(u_k)$$

$$\cdot du_1 \dots du_k \in \bigotimes_n F.$$

Then we call  $\phi$  a normal generalized random variable.

**Remark:** From the definition  $\phi$  can be written as

$$\phi \equiv \int_{\mathbb{R}^k} f(u_1, \dots, u_k) : \langle \cdot, \theta(u_1) \rangle^{n_1} \dots \langle \cdot, \theta(u_k) \rangle^{n_k} : du_1 \dots du_k.$$

**Example 1.**

Suppose  $\phi = : \langle \cdot, h_1 \rangle \dots \langle \cdot, h_n \rangle :$  is an ordinary random variable in  $K_n$ , i.e.  $U(\phi)(\xi) = (\xi, h_1) \dots (\xi, h_n)$ . Then

$$U(\partial_{\theta(t)} \phi) = \sum_{j=1}^n \langle \xi, h_1 \rangle \dots \langle \xi, h_n \rangle \langle \theta(t), h_j \rangle$$

$$U(\partial_{\theta(t)}^* \partial_{\theta(t)} \phi) = \langle \xi, \theta(t) \rangle \sum_{j=1}^n \langle \xi, h_1 \rangle \dots \langle \xi, h_n \rangle \langle \theta(t), h_j \rangle$$

$$U(\partial_{\theta(t)}^2 \phi) = \sum_{\substack{i,j \\ i \neq j}} \langle \xi, h_1 \rangle \dots \langle \xi, h_n \rangle \langle \theta(t), h_i \rangle \langle \theta(t), h_j \rangle.$$

Therefore

$$\Delta_G \phi = \int_{\mathbb{R}} \partial_{\theta(t)}^2 \phi dt = \sum_{\substack{i,j \\ i \neq j}} \langle \cdot, h_1 \rangle \dots \langle \cdot, h_n \rangle \langle h_i, h_j \rangle$$

$$\Delta_B \phi = -n\phi$$

$$\Delta_L \phi = 0$$

$$\Delta_V \phi = \Delta_G \phi.$$

**Example 2.**

Suppose  $\phi(x) = \int_{\mathbb{R}} f(u) : \langle \cdot, \theta(u) \rangle^n : du$  is a normal generalized random variable in  $K_n^{(-)}$ , i.e.,  $\int_{\mathbb{R}} f(u) \theta(u) \underset{n \text{ times}}{\otimes} \dots \underset{n \text{ times}}{\otimes} \theta(u) du \in \underset{n}{\otimes} F$ . Then

$$U(\partial_{\theta(t)} \phi)(\xi) = n \int_{\mathbb{R}} f(u) \langle \xi, \theta(u) \rangle^{n-1} \langle \theta(t), \theta(u) \rangle du$$

$$U(\partial_{\theta(t)}^* \partial_{\theta(t)} \phi)(\xi) = n \int_{\mathbb{R}} f(u) \langle \xi, \theta(u) \rangle^{n-1} \langle \xi, \theta(t) \rangle \langle \theta(t), \theta(u) \rangle du$$

$$U(\partial_{\Theta(t)}^2 \phi)(\xi) = n(n-1) \int f(u) \langle \xi, \Theta(u) \rangle^{n-2} \langle \Theta(t), \Theta(u) \rangle \langle \Theta(t), \Theta(u) \rangle du.$$

Therefore,

$$\Delta_G : \text{ does not exist}$$

$$\Delta_B = -n\phi \quad \text{since} \quad \int_{\mathbb{R}} \langle \xi, \Theta(t) \rangle \langle \Theta(t), \Theta(u) \rangle dt = \langle \xi, \Theta(u) \rangle$$

$$\Delta_L = n(n-1) \int f(u) \langle \cdot, \Theta(u) \rangle^{n-2} du$$

$$\Delta_V = 0.$$

### Example 3

$\phi = : \exp \left[ c \sum_{j=1}^{\infty} \langle \cdot, e_j \rangle^2 \right] :$   $c \neq \frac{1}{2}$  is a generalized Gaussian random variable:

$$(U\phi)(\xi) = \exp \left[ \frac{c}{1-2c} \|\xi\|^2 \right].$$

Therefore

$$(U\phi)'(\xi; \Theta(t)) = \frac{2c}{1-2c} \langle \xi, \Theta(t) \rangle \exp \left[ \frac{c}{1-2c} (\xi, \xi) \right] = \frac{2c}{1-2c} U(\partial_{\Theta(t)}^* \phi).$$

Thus

$$\begin{aligned} U(\partial_{\Theta(t)}^2 \phi) &= (U''\phi)(\xi; \Theta(t), \Theta(t)) \\ &= \left( \frac{2c}{1-2c} \right)^2 \langle \xi, \Theta(t) \rangle \langle \xi, \Theta(t) \rangle \exp \left[ \frac{c}{1-2c} (\xi, \xi) \right] \\ &\quad + \left( \frac{2c}{1-2c} \right) \langle \Theta(t), \Theta(t) \rangle \exp \left[ \frac{c}{1-2c} (\xi, \xi) \right] \end{aligned}$$

$$U(\partial_{\Theta(t)}^* \partial_{\Theta(t)} \phi)(\xi) = \frac{2c}{1-2c} \langle \xi, \Theta(t) \rangle \langle \xi, \Theta(t) \rangle \exp \left[ \frac{c}{1-2c} (\xi, \xi) \right].$$

Therefore,

$$\Delta_G \text{ does not exist}$$

$$\Delta_B \phi = \frac{2c}{2c-1} : \sum_{j=1}^{\infty} \langle \cdot, e_j \rangle^2 : \phi(x) \quad \text{where} \quad U\left( : \sum_{j=1}^{\infty} \langle \cdot, e_j \rangle^2 : \right)(\xi) = (\xi, \xi)_H$$

$$\Delta_L \phi = \frac{2c}{1-2c} \phi(x) \int_{\mathbb{R}^1} dt$$

$$\Delta_V \phi = \left(\frac{2c}{1-2c}\right)^2 : \sum_{j=1}^{\infty} \langle \cdot, e_j \rangle^2 : \phi(x) = \left(\frac{2c}{2c-1}\right) \Delta_B \phi.$$

where  $\int_{\mathbb{R}^1} dt$  is a non-standard real number. (Actually we avoid  $\int_{\mathbb{R}^1} dt$  by considering  $\phi = : \exp c \int_T \langle \cdot, \theta(t) \rangle \langle \cdot, \theta(t) \rangle dt :$ )

**Remark:** The generalized Gaussian random variable  $\phi$  in Example 3 is an eigenfunction of  $\Delta_L$ .

## CHAPTER 4. Fourier transform on $(L^2)^-$

### §1. Fourier transform.

A Fourier transform introduced by H.-H. Kuo is very useful for casual calculus. This Fourier transform carries  $\dot{B}(t)$ -differentiation into multiplication by  $i\dot{b}(t)$ . Such a Fourier transform can be defined in the space  $(L^2)^-$  of generalized random variables. We will use  $x$  and  $y$  to denote variables in the domain and range spaces respectively.

#### Renormalization of $\exp[-i\langle x, y \rangle]$

If  $x \in B$  and  $y \in H$ , then  $\langle x, y \rangle$  is an ordinary random variable and  $\exp[-i\langle x, y \rangle]$  is also an ordinary random variable. But if  $x \in B$ ,  $y \in B \setminus H$ , then  $\langle x, y \rangle$  is not a random variable in the ordinary sense. So renormalization is required to define  $:\exp[-i\langle x, y \rangle]:_y$ . Let

$$\phi_N(y) = \exp\left[-i \sum_{j=1}^N \langle x, e_j \rangle \langle y, e_j \rangle\right].$$

Then obviously  $\phi_N(y) \in (L^2)^+$ . Now

$$\begin{aligned} S(\phi_N(y))(\xi) &= \int_B \exp\left[-i \sum_{j=1}^N \langle y + \xi, e_j \rangle \langle x, e_j \rangle\right] d\mu(y) \\ &= \int_B \exp\left[-i \sum_{j=1}^N (\langle x, e_j \rangle \langle y, e_j \rangle + \langle \xi, e_j \rangle \langle x, e_j \rangle)\right] d\mu(y) \end{aligned}$$

$$\begin{aligned}
&= \exp\left[-1 \sum_{j=1}^N (\langle \xi, e_j \rangle \langle x, e_j \rangle)\right] \int_B \exp\left[-1 \sum_{j=1}^N (\langle x, e_j \rangle \langle y, e_j \rangle)\right] d\mu(y) \\
&= \exp\left[-1 \sum_{j=1}^N (\langle \xi, e_j \rangle \langle x, e_j \rangle)\right] \prod_{j=1}^N \int_B \exp[-1 \langle x, e_j \rangle \langle y, e_j \rangle] d\mu(y) \\
&= \exp\left[-1 \sum_{j=1}^N (\langle \xi, e_j \rangle \langle x, e_j \rangle)\right] \prod_{j=1}^N e^{-\frac{1}{2} |\langle x, e_j \rangle e_j|^2} \\
&= \exp\left[-1 \sum_{j=1}^N (\langle \xi, e_j \rangle \langle x, e_j \rangle)\right] \prod_{j=1}^N e^{-\frac{1}{2} |\langle x, e_j \rangle|^2}
\end{aligned}$$

Therefore

$$S\left(e^{\sum_{j=1}^N \frac{1}{2} \langle x, e_j \rangle^2} \phi_N(y)\right)(\xi) = \exp\left[-1 \sum_{j=1}^N (\langle \xi, e_j \rangle \langle x, e_j \rangle)\right]$$

and

$$\exp\left[-1 \sum_{j=1}^N (\langle \xi, e_j \rangle \langle x, e_j \rangle)\right] \longrightarrow \exp[-1 \langle x, \xi \rangle] \quad \text{as } N \rightarrow \infty.$$

**Note:**  $\exp[-1 \langle x, \xi \rangle] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle x, \xi \rangle^k$  and

$\frac{(-1)^k}{k!} x \hat{\otimes} \dots \hat{\otimes}_k x \in \hat{\otimes}_k F$  a.e.  $\mu(x)$  (Here  $x \in B$ . But we can assume

$x \in F$  a.e.  $(\mu)$  since  $H \subset F$  is an abstract Wiener space). Therefore,

we can have  $(L^2)^{-}$  limit of  $e^{\frac{1}{2} \sum_{j=1}^N \langle x, e_j \rangle^2} \phi_N(y)$ , denoted by  $:\exp[-1 \langle x, y \rangle]:_y$ ; and

$$U(:\exp[-1 \langle x, y \rangle]:_y)(\xi) = \exp[-1 \langle \xi, x \rangle].$$

We also see that

$$\begin{aligned}
e^{\frac{1}{2} \sum_{j=1}^N \langle x, e_j \rangle^2} \phi_N(y) &\xrightarrow{N \rightarrow \infty} e^{\frac{1}{2} \sum_{j=1}^{\infty} \langle x, e_j \rangle^2} \exp\left[-1 \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle y, e_j \rangle\right] \\
&= e^{\frac{1}{2} \sum_{j=1}^{\infty} \langle x, e_j \rangle^2} \exp[-1 \langle x, y \rangle].
\end{aligned}$$

Formally, this can be written as



$$e^{\frac{1}{2}\|x\|_H^2} \exp[-i\langle x, y \rangle].$$

Here it is obvious  $e^{\frac{1}{2}\|x\|_H^2} = \infty$  in the ordinary sense. But we can regard  $e^{\frac{1}{2} \sum_{j=1}^{\infty} \langle x, e_j \rangle^2} = e^{\frac{1}{2}\|x\|_H^2}$  in the non-standard real number. Therefore,

$$:\exp[-i\langle x, y \rangle]:_y \equiv e^{\frac{1}{2}\|x\|_H^2} \exp[-i\langle x, y \rangle].$$

#### Definition 4.1

The renormalization of  $\exp[-i\langle x, y \rangle]$  with respect to the  $y$ -variable, denoted by  $:\exp[-i\langle x, y \rangle]:_y$ , is the generalized random variable such that  $U(:\exp[-i\langle x, y \rangle]:_y)(\xi) = \exp[-i\langle \xi, x \rangle]$ .

Note: It is easy to see that  $y \rightarrow :\exp[-i\langle x, y \rangle]:_y$  is a measurable map from  $B$  into  $(L^2)_y^-$ .

#### Lemma 4.1.

If  $\phi(y) \in K_{n,y}^{(+)}$  is represented by  $k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^*$  with  $k_i^* \in F^*$  for  $i = 1, 2, \dots, n$ , then

$$\langle :\exp[-i\langle x, y \rangle]:_y, \phi \rangle = (-1)^n \langle x, k_1^* \rangle \dots \langle x, k_n^* \rangle$$

and

$$\langle :\exp[-i\langle x, y \rangle]:_y, \phi(y) \rangle \in K_{n,x}^{(+)}.$$

Proof: Observe that

$$U(:\exp[-i\langle x, y \rangle]:_y)(\xi) = \exp[-i\langle \xi, x \rangle] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle \xi, x \rangle^k$$

and

$$\frac{(-1)^n}{n!} x \hat{\otimes} \dots \hat{\otimes}_{n \text{ times}} x \in \hat{\otimes}_n F \quad \text{a.e.} \quad (u).$$

Therefore

$$\begin{aligned} & \langle : \exp[-i\langle x, y \rangle] :_y, \phi \rangle \\ &= n! \langle \frac{(-1)^n}{n!} x \hat{\otimes} \dots \hat{\otimes} x, k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^* \rangle \\ &= (-1)^n \langle x \hat{\otimes} \dots \hat{\otimes} x, k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^* \rangle \\ &= (-1)^n \langle x, k_1^* \rangle \dots \langle x, k_n^* \rangle. \end{aligned}$$

Also  $k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^* \in \hat{\otimes}_n F^*$  shows that  $\langle : \exp[-i\langle x, y \rangle] :_y, \phi \rangle \in K_{n,x}^{(+)}$ .  
(Q.E.D.)

#### Lemma 4.2

For any  $\phi(y) \in (L^2_y)^+$ ,

$$\partial_{\langle x, \theta(t) \rangle}^* (\langle : \exp[-i\langle x, y \rangle] :_y, \phi \rangle) = i \langle : \exp[-i\langle x, y \rangle] :_y, \langle y, \theta(t) \rangle \phi \rangle.$$

Proof: It is sufficient to prove this for  $\phi(y) \in K_{n,y}^{(+)}$ . Let us assume  $\phi$  to be represented by  $k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^* \in \hat{\otimes}_n F^*$ . Then by Lemma 4.1,

$$\langle : \exp[-i\langle x, y \rangle] :_y, \phi(y) \rangle \in K_{n,x}^{(+)}$$

and

$$\langle : \exp[-i\langle x, y \rangle] :_y, \phi(y) \rangle = (-1)^n \langle x, k_1^* \rangle \dots \langle x, k_n^* \rangle$$

i.e.,  $\langle : \exp[-i\langle x, y \rangle] :_y, \phi(y) \rangle$  is represented by

$$(-1)^n k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^*.$$

Now by Theorem 3.3 (1),  $\partial_{\langle x, \theta(t) \rangle}^* (\langle : \exp[-i\langle x, y \rangle] :_y, \phi \rangle)$  is represented by

$$(-1)^n \sum_{j=1}^n \langle k_j^*, \theta(t) \rangle k_1^* \hat{\otimes} \dots \hat{\otimes}_{(j)} k_n^*$$

On the other hand,

$$\begin{aligned}\langle y, \theta(t) \rangle \phi &= \partial \langle y, \theta(t) \rangle \phi + \partial^* \langle y, \theta(t) \rangle \phi \\ &= \sum_{j=1}^n \langle k_j^*, \theta(t) \rangle \langle k_1^*, y \rangle \dots \langle k_n^*, y \rangle + \langle k_1^*, y \rangle \dots \langle k_n^*, y \rangle \langle \theta(t), y \rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}& \langle : \exp[-i\langle x, y \rangle] :_y, \langle y, \theta(t) \rangle \phi \rangle \\ &= \langle : \exp[-i\langle x, y \rangle] :_y, \partial \langle y, \theta(t) \rangle \phi + \partial^* \langle y, \theta(t) \rangle \phi \rangle \\ &= \langle : \exp[-i\langle x, y \rangle] :_y, \partial \langle y, \theta(t) \rangle \phi \rangle + \langle : \exp[-i\langle x, y \rangle] :_y, \partial^* \langle y, \theta(t) \rangle \phi \rangle \\ &= (n-1)! (-1)^{n-1} \cdot \frac{1}{(n-1)!} \times \hat{\otimes}_{(n-1) \text{ times}} x, \sum_{j=1}^n \langle k_j^*, \theta(t) \rangle k_1^* \hat{\otimes} \dots \hat{\otimes}_{(j)} k_n^* \rangle \\ &+ (n+1)! (-1)^{n-1} \cdot \frac{1}{(n+1)!} \times \hat{\otimes}_{(n+1) \text{ times}} x, k_1^* \hat{\otimes} \dots \hat{\otimes} k_n^* \otimes \theta(t) \rangle \\ &= (-1)^{n-1} \sum_{j=1}^n \langle k_j^*, \theta(t) \rangle \langle k_1^*, x \rangle \dots \langle k_{j-1}^*, x \rangle \langle k_{j+1}^*, x \rangle \dots \langle k_n^*, x \rangle \\ &+ (-1)^{n+1} \langle x, \theta(t) \rangle \langle x, k_1^* \rangle \dots \langle x, k_n^* \rangle,\end{aligned}$$

and so

$$\begin{aligned}& i \langle : \exp[-i\langle x, y \rangle] :_y, \langle y, \theta(t) \rangle \phi \rangle \\ &= -\partial \langle x, \theta(t) \rangle (\langle : \exp[-i\langle x, y \rangle] :_y, \phi \rangle) + \langle x, \theta(t) \rangle \langle : \exp[-i\langle x, y \rangle] :_y, \phi \rangle \\ &= \partial^* \langle x, \theta(t) \rangle (\langle : \exp[-i\langle x, y \rangle] :_y, \phi \rangle). \quad (\text{Q.E.D.})\end{aligned}$$

#### **Definition 4.2**

For  $\phi(x) \in (L^2_x)^{-}$ , the Fourier transform  $\hat{\phi}(y)$  of  $\phi$  is defined by

$$\hat{\phi}(y) = \int_B : \exp[-i\langle x, y \rangle] :_y \phi(x) d\mu(x).$$

**Remark:** The above definition can be interpreted as follows:

$$\hat{\phi}(y) \in (L^2)_y^-$$

satisfies

$$\begin{aligned} \langle \hat{\phi}(y), \psi(y) \rangle &= \int_B \langle : \exp[-i\langle x, y \rangle] :_y, \phi(x) \rangle \psi(y) d\mu(y) \\ &= \int_B \langle : \exp[-i\langle x, y \rangle] :_y, \psi(y) \rangle \phi(x) d\mu(x) \\ &= \langle \phi, \langle : \exp[-i\langle x, y \rangle] :_y, \psi \rangle \rangle \\ &\text{for } \psi(y) \in (L^2)_y^+. \end{aligned}$$

(Since  $\langle : \exp[-i\langle x, y \rangle] :_y, \psi \rangle \in (L^2)_x^+$  by Lemma 4.1, this is well-defined).

**Theorem 4.1.**

$$(\partial_{\langle x, \theta(t) \rangle} \phi(x))^{\wedge}(y) = i\langle y, \theta(t) \rangle \hat{\phi}(y).$$

**Remark:** This theorem states that the Fourier transform carries  $\langle x, \theta(t) \rangle$ -differentiation into multiplication by  $i\langle y, \theta(t) \rangle$ .

**Proof:** Let  $\psi(y) \in (L^2)_y^+$ . Then by Lemma 4.2

$$\partial_{\langle x, \theta(t) \rangle}^* (\langle : \exp[-i\langle x, y \rangle] :_y, \psi \rangle) = i \langle : \exp[-i\langle x, y \rangle] :_y, \langle y, \theta(t) \rangle \psi \rangle.$$

Therefore, for all  $\psi \in (L^2)_y^+$

$$\begin{aligned} \langle (\partial_{\langle x, \theta(t) \rangle} \phi(x))^{\wedge}, \psi \rangle &= \langle \partial_{\langle x, \theta(t) \rangle} \phi(x), \langle : \exp[-i\langle x, y \rangle] :_y, \psi \rangle \rangle \\ &= \langle \phi, \partial_{\langle x, \theta(t) \rangle}^* \langle : \exp[-i\langle x, y \rangle] :_y, \psi \rangle \rangle \\ &= \langle \phi, i \langle : \exp[-i\langle x, y \rangle] :_y, \langle y, \theta(t) \rangle \psi \rangle \rangle \end{aligned}$$

$$= \langle \hat{\phi}, 1_{\langle y, \theta(t) \rangle} \psi \rangle$$

$$= \langle 1_{\langle y, \theta(t) \rangle} \hat{\phi}, \psi \rangle$$

Therefore,  $(\partial_{\langle x, \theta(t) \rangle} \phi(x))^{\wedge}(y) = 1_{\langle y, \theta(t) \rangle} \hat{\phi}(y)$ . (Q.E.D.)

**Theorem 4.2**

For  $\phi(x) \in (L^2)_x^-$

$$U(\hat{\phi}(y))(n) = \exp(-\frac{1}{2}\|n\|^2)U(\phi(x))(-in).$$

**Proof:** By definition

$$\begin{aligned} U(\hat{\phi}(y))(n) &= \int_{\mathbb{B}} \hat{\phi}(y + n) d\mu(y) \\ &= \int_{\mathbb{B}} \int_{\mathbb{B}} : \exp[-i\langle x, y + n \rangle] :_y \phi(x) d\mu(x) d\mu(y) \\ &= \int_{\mathbb{B}} \int_{\mathbb{B}} : \exp[-i\langle x, y + n \rangle] :_y d\mu(y) \phi(x) d\mu(x) \\ &= \int_{\mathbb{B}} U(: \exp[-i\langle x, y \rangle] :_y)(n) \phi(x) d\mu(x) \\ &= \int_{\mathbb{B}} \exp[-i\langle n, x \rangle] \phi(x) d\mu(x). \end{aligned}$$

Now

$$U(\hat{\phi}(y))(in) = \int_{\mathbb{B}} \exp[\langle n, x \rangle] \phi(x) d\mu(x).$$

Let  $x = y + n$ , and use  $d\mu(y + n) = e^{-\frac{1}{2}(\|n\|^2) - \langle n, y \rangle} d\mu(y)$ , then the above is equal to

$$\begin{aligned} &\int_{\mathbb{B}} \exp[\langle n, y + n \rangle] \phi(y + n) e^{-\frac{1}{2}(\|n\|^2) - \langle n, y \rangle} d\mu(y) \\ &= \exp[\frac{1}{2}\|n\|^2] \int_{\mathbb{B}} \phi(y + n) d\mu(y) \\ &= \exp[\frac{1}{2}\|n\|^2] U\phi(n). \end{aligned} \quad (\text{Q.E.D.})$$

**Corollary 4.2**

If  $\phi(x) \in (L^2)_x^-$ , then

$$U(\hat{\phi}(y))(n) = \int_B e^{-1\langle x, n \rangle} \phi(x) d\mu(x) \quad \text{for } n \in B^*$$

**Theorem 4.3**

Suppose  $\phi(x) \in K_{n,x}^{(-)}$  is represented by  $k_1 \hat{\otimes} \dots \hat{\otimes} k_n$ , where  $k_i \in F$  for  $i = 1, \dots, n$ . Then

$$U(\hat{\phi}(y))(n) = e^{\frac{1}{2}\|n\|^2} (-1)^n \langle n, k_1 \rangle \dots \langle n, k_n \rangle.$$

**Proof:** From Corollary 4.2 we see that

$$U(\hat{\phi}(y))(n) = \langle e^{-1\langle x, n \rangle}, \phi(x) \rangle \quad \text{for } \phi \in (L^2)^-.$$

Also

$$\begin{aligned} U(e^{-1\langle x, n \rangle})(\xi) &= \int_B e^{-1\langle x+\xi, n \rangle} d\mu(x) \\ &= e^{-1\langle \xi, n \rangle} \int_B e^{-1\langle x, n \rangle} d\mu(x) \\ &= e^{-1\langle \xi, n \rangle} \cdot e^{-\frac{1}{2}\|n\|^2} \\ &= e^{-\frac{1}{2}\|n\|^2} \sum_{n=0}^{\infty} \frac{(-1)^n \langle \xi, n \rangle^n}{n!} \end{aligned}$$

Therefore,

$$\begin{aligned} \langle e^{-1\langle x, n \rangle}, \phi(x) \rangle &= n! \langle e^{-\frac{1}{2}\|n\|^2} \frac{(-1)^n}{n!} n \hat{\otimes} \dots \hat{\otimes} n, k_1 \hat{\otimes} \dots \hat{\otimes} k_n \rangle \\ &= e^{-\frac{1}{2}\|n\|^2} (-1)^n \langle n \hat{\otimes} \dots \hat{\otimes} n, k_1 \hat{\otimes} \dots \hat{\otimes} k_n \rangle \\ &= (-1)^n e^{-\frac{1}{2}\|n\|^2} \langle n, k_1 \rangle \dots \langle n, k_n \rangle. \quad (\text{Q.E.D.}) \end{aligned}$$

**Theorem 4.4**

Let  $\phi \in (L^2)^-$  and

$$F(u_1, \dots, u_n) \equiv \int_B \langle x, \Theta(u_1) \rangle \langle x, \Theta(u_2) \rangle \dots \langle x, \Theta(u_n) \rangle \phi(x) d\mu(x) \in \hat{L}^2(\mathbb{R}^n).$$

Then for any  $\psi \in K_{n,y}^{(+)}$  represented by  $G(u_1, \dots, u_n)$  with  $G(u_1, \dots, u_n) \in \hat{L}^2(\mathbb{R}^n)$ , we have

$$\langle \hat{\phi}, \psi \rangle = (-1)^n \int_{\mathbb{R}^n} F(u_1, \dots, u_n) G(u_1, \dots, u_n) du_1 \dots du_n.$$

**Remark:** Suppose  $\psi \in K_n$  has the representation  $G(u_1, \dots, u_n)$  then

$$\begin{aligned} \psi(\cdot) &= \int_{\mathbb{R}^n} G(u_1, \dots, u_n) d\beta(u_1) \dots d\beta(u_n) \\ &= \int_{\mathbb{R}^n} G(u_1, \dots, u_n) : \langle \cdot, \Theta(u_1) \rangle \langle \cdot, \Theta(u_2) \rangle \dots \langle \cdot, \Theta(u_n) \rangle : du_1, \dots, du_n \end{aligned}$$

$$\text{and } \|\psi\|_{(L^2)} = \sqrt{n!} \|G\|_{L^2(\mathbb{R}^n)}.$$

(Proof)

$$\begin{aligned} U(\hat{\phi}(y))(\eta) &= \int_B e^{-1 \langle x, \eta \rangle} \phi(x) d\mu(x) \\ &= \int_B \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle x, \eta \rangle^n \phi(x) d\mu(x) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_B \left[ \int_{\mathbb{R}^n} \langle x, \Theta(u_1) \rangle \dots \langle x, \Theta(u_n) \rangle \right. \\ &\quad \left. \cdot \langle \eta, \Theta(u_1) \rangle \dots \langle \eta, \Theta(u_n) \rangle du_1 \dots du_n \right] \phi(x) d\mu(x) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \left[ \int_B \langle x, \Theta(u_1) \rangle \dots \langle x, \Theta(u_n) \rangle \phi(x) d\mu(x) \right] \\ &\quad \cdot \langle \eta, \Theta(u_1) \rangle \dots \langle \eta, \Theta(u_n) \rangle du_1 \dots du_n. \end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \hat{\phi}, \psi \rangle &= n! \langle \frac{(-1)^n}{n!} \int_B \langle x, \theta(u_1) \rangle \dots \langle x, \theta(u_n) \rangle \phi(x) d\mu(x), \mathbb{E}(u_1, \dots, u_n) \rangle_{L^2(\mathbb{R}^n)} \\
&= (-1)^n \int_{\mathbb{R}^n} G(u_1, \dots, u_n) F(u_1, \dots, u_n) du_1 \dots du_n
\end{aligned}$$

where

$$F(u_1, \dots, u_n) \equiv \int_B \langle x, \theta(u_1) \rangle \dots \langle x, \theta(u_n) \rangle \phi(x) d\mu(x). \quad (\text{Q.E.D.})$$

#### **Theorem 4.5**

Suppose  $\phi \in K_{m,x}$  and  $\psi \in K_{n,y}$  are represented by  $F(u_1, \dots, u_m)$  and  $G(u_1, \dots, u_n)$ , respectively. Then

$$\begin{aligned}
\langle \hat{\phi}, \psi \rangle &= 0 && \text{if } m > n \\
&= (-1)^n n! \langle F, G \rangle_{L^2(\mathbb{R}^n)} && \text{if } m = n
\end{aligned}$$

**Proof:** By Theorem 4.4

$$\begin{aligned}
\langle \hat{\phi}, \psi \rangle &= (-1)^n \int_{\mathbb{R}^n} \left[ \int_B \langle x, \theta(u_1) \rangle \dots \langle x, \theta(u_n) \rangle \phi(x) d\mu(x) \right] \\
&\quad G(u_1, \dots, u_n) du_1 \dots du_n \\
&= (-1)^n \int_B \left[ \int_{\mathbb{R}^n} G(u_1, \dots, u_n) \langle x, \theta(u_1) \rangle \dots \langle x, \theta(u_n) \rangle du_1 \dots du_n \right] \\
&\quad \cdot \phi(x) d\mu(x).
\end{aligned}$$

Now let  $P_n$  be the orthogonal projection of  $L^2(B)$  onto  $K_n$ . Then

$$P_m \left( \int_{\mathbb{R}^n} G(u_1, \dots, u_n) \langle x, \theta(u_1) \rangle \dots \langle x, \theta(u_n) \rangle du_1 \dots du_n \right) = 0 \quad \text{if } m > n$$

and

$$\begin{aligned}
&P_m \left( \int_{\mathbb{R}^n} G(u_1, \dots, u_n) \langle x, \theta(u_1) \rangle \dots \langle x, \theta(u_n) \rangle du_1 \dots du_n \right) \\
&= \int_{\mathbb{R}^n} G(u_1, \dots, u_n) \langle x, \theta(u_1) \rangle \dots \langle x, \theta(u_n) \rangle du_1 \dots du_n
\end{aligned}$$



$$= \int_{\mathbb{R}^n} G(u_1, \dots, u_n) d\beta(u_1) \dots d\beta(u_n) \quad \text{if } m = n.$$

$$\text{Therefore } \langle \hat{\phi}, \psi \rangle = \begin{cases} 0 & \text{if } m > n \\ (-1)^n n! \langle F, G \rangle & \text{if } m = n. \end{cases} \quad (\text{Q.E.D.})$$

**Example 1.** Let  $\phi(x) \equiv 1$ . Then

$$U(\hat{\phi}(y))(n) = \int_B e^{-i\langle x, n \rangle} d\mu(x)$$

$$= \int_B e^{-i\langle x, n \rangle} d\mu(x)$$

$$= e^{-\frac{1}{2}\|n\|^2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n! 2^n} (n, n)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n! 2^n} \int_{\mathbb{R}^n} \langle n, \theta(u_1) \rangle^2 \langle n, \theta(u_2) \rangle^2 \dots \langle n, \theta(u_n) \rangle^2 du_1 \dots du_n.$$

Therefore

$$\hat{\phi}(y) = \sum_{n=0}^{\infty} \frac{1}{n! (2)^n} \int_{\mathbb{R}^n} \langle y, \theta(u_1) \rangle^2 \dots \langle y, \theta(u_n) \rangle^2 du_1 \dots du_n.$$

**Example 2.** Suppose  $\phi(x) = \langle x, \xi_0 \rangle$  with  $\xi_0 \in B^*$ . Then, by Theorem 4.3,

$$U(\hat{\phi}(y))(n) = e^{-\frac{1}{2}\|n\|^2} \cdot (-i) \langle n, \xi_0 \rangle$$

Therefore,

$$\hat{\phi}(y) = \sum_{n=0}^{\infty} \frac{-i}{n! 2^n} \int_{\mathbb{R}^n} \langle y, \theta(u_1) \rangle^2 \dots \langle y, \theta(u_n) \rangle^2 \langle y, \xi_0 \rangle du_1 \dots du_n.$$

**Example 3.** Suppose

$$\phi(x) = \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \langle x, \theta(u_1) \rangle^{n_1} \dots \langle x, \theta(u_k) \rangle^{n_k} du_1 \dots du_k$$

is normal generalized random variable with  $\sum_{j=1}^k n_j = n$  then

$$\hat{\phi}(y) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! 2^j} \int_{\mathbb{R}^{k+j}} F(u_1, \dots, u_k) : \langle y, \theta(u_1) \rangle^{n_1} \dots \langle y, \theta(u_k) \rangle^{n_k} \\ \langle y, \theta(u_{k+1}) \rangle^2 \dots \langle y, \theta(u_{k+j}) \rangle^2 du_1 \dots du_{k+j}.$$

This can be verified as follows. We know

$$U(\phi(x))(\xi) = \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \langle \xi, \theta(u_1) \rangle^{n_1} \dots \langle \xi, \theta(u_k) \rangle^{n_k} du_1 \dots du_k.$$

Therefore,

$$U(\phi(x))(-in) = (-1)^n \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \\ \cdot \langle n, \theta(u_1) \rangle^{n_1} \dots \langle n, \theta(u_k) \rangle^{n_k} du_1 \dots du_k.$$

Then

$$U(\hat{\phi}(y))(n) = e^{+\frac{1}{2}\|n\|^2} U(\phi(x))(-in) \quad \text{by Theorem 4.2} \\ = e^{\frac{1}{2}\|n\|^2} \cdot (-1)^n \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \langle n, \theta(u_1) \rangle^{n_1} \dots \langle n, \theta(u_k) \rangle^{n_k} du_1 \dots du_k \\ = \sum_{j=0}^{\infty} \frac{1}{j! 2^j} (n, n)^j \cdot (-1)^n \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \\ \cdot \langle n, \theta(u_1) \rangle^{n_1} \dots \langle n, \theta(u_k) \rangle^{n_k} du_1 \dots du_k \\ = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! 2^j} \int_{\mathbb{R}^{k+j}} F(u_1, \dots, u_k) \langle n, \theta(u_1) \rangle^{n_1} \dots \langle n, \theta(u_k) \rangle^{n_k} \\ \cdot \langle n, \theta(u_{k+1}) \rangle^2 \dots \langle n, \theta(u_{k+j}) \rangle^2 du_1 \dots du_{k+j}.$$

Therefore,

$$\hat{\phi}(y) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! 2^j} \int_{\mathbb{R}^{k+j}} F(u_1, \dots, u_k) : \langle y, \theta(u_1) \rangle^{n_1} \dots \langle y, \theta(u_k) \rangle^{n_k} \\ \cdot \langle y, \theta(u_{k+1}) \rangle^2 \dots \langle y, \theta(u_{k+j}) \rangle^2 du_1 \dots du_{k+j}.$$

## §2. Relation between the Fourier transform and the Levy Laplacian.

There are several operators that may be viewed as infinite dimensional analogs of the ordinary Laplacian in  $\mathbb{R}^n$ . T. Hida and K. Saito [7] have obtained a relation between Kuo's Fourier transform and the Levy Laplacian in white noise calculus. Similarly we can obtain a similar relation between the Fourier transform given in Definition 4.2 and the Levy Laplacian  $\Delta_L$ .

Now let us specify the domain of  $\Delta_L$ . Let  $W$  be the vector space spanned by the normal random variables given in Definition 3.5. Define a norm  $\|\cdot\|_e$  in  $W$  as follows:

$$\|\phi\|_e^2 = \|\phi\|_{(L^2)^-}^2 + \int \|\partial_{\theta(t)} \phi\|_{(L^2)^-}^2 dt + \int \|\partial_{\theta(t)}^2 \phi D(t)\|_{(L^2)^-}^2 dt.$$

We see easily  $\|\phi\|_e < \infty$  for  $\phi \in W$ . The completion of  $W$  with respect to the norm  $\|\cdot\|_e$  will be denoted by  $\bar{W}$  and will be taken to be the domain of the Levy Laplacian  $\Delta_L$ .

### Theorem 4.6

For  $\phi \in \bar{W}$

- i)  $\|\Delta_L \phi\|_{(L^2)^-} \leq \|\phi\|_e$
- ii)  $\Delta_L \phi = \int \partial_{\theta(t)}^2 \phi(D(t)dt).$
- iii)  $\Delta_L \phi = 0$  for  $\phi \in (L^2).$

Proof: (i) It is enough to show this for  $\phi \in W$ . Then it is obvious since  $\|\Delta_L \phi\|_{(L^2)^-} = \|\int \partial_{\theta(t)}^2 \phi D(t)dt\|_{(L^2)^-} < \int \|\partial_{\theta(t)}^2 \phi D(t)\|_{(L^2)^-} dt.$

(ii) is mentioned in Remark 1 after Definition 3.4;

(iii) is proved in Example 1 in §2, Chapter 3.

**Theorem 4.7**

$$(\Delta_L - 1)\hat{\phi} = -(\Delta_L \phi)^\wedge \quad \text{for } \phi \in \overline{W}.$$

**Proof:** It is sufficient to show that the theorem is true for  $\phi$  of the form in Definition 3.5. Then from Example 3 in §1

$$\begin{aligned} U(\hat{\phi}(y))(\eta) &= e^{\frac{1}{2}\|\eta\|^2} U(\phi(x))(-i\eta) \\ &= e^{\frac{1}{2}\|\eta\|^2} \cdot (-1)^n \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \langle \eta, \theta(u_1) \rangle^{n_1} \dots \langle \eta, \theta(u_k) \rangle^{n_k} du_1 \dots du_k \\ U(\Delta_L \hat{\phi})(\eta) &= e^{\frac{1}{2}\|\eta\|^2} (-1)^n \cdot \sum_{\ell=1}^k \eta_\ell (\eta_\ell - 1) \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \\ &\quad \cdot \langle \eta, \theta(u_1) \rangle^{n_1} \dots \langle \eta, \theta(u_\ell) \rangle^{n_\ell - 2} \dots \langle \eta, \theta(u_k) \rangle^{n_k} du_1 \dots du_k \\ &+ (-1)^n \cdot e^{\frac{1}{2}\|\eta\|^2} \cdot \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \langle \eta, \theta(u_1) \rangle^{n_1} \dots \langle \eta, \theta(u_k) \rangle^{n_k} du_1 \dots du_k. \end{aligned}$$

On the other hand,

$$\begin{aligned} U(\Delta_L \phi(x))(\eta) &= \sum_{\ell=1}^k (\eta_\ell)(\eta_\ell - 1) \int_{\mathbb{R}^k} F(u_1, \dots, u_k) \\ &\quad \cdot \langle \eta, \theta(u_1) \rangle^{n_1} \dots \langle \eta, \theta(u_\ell) \rangle^{n_\ell - 2} \dots \langle \eta, \theta(u_k) \rangle^{n_k} du_1 \dots du_k \end{aligned}$$

Therefore,

$$\begin{aligned} U(\Delta_L \phi(x))^\wedge(\eta) &= e^{\frac{1}{2}\|\eta\|^2} (-1)^{n-2} \cdot \sum_{\ell=1}^k (\eta_\ell)(\eta_\ell - 1) \int_{\mathbb{R}^k} F(u_1 \dots u_k) \\ &\quad \cdot \langle \eta, \theta(u_1) \rangle^{n_1} \langle \eta, \theta(u_\ell) \rangle^{n_\ell - 2} \dots \langle \eta, \theta(u_k) \rangle^{n_k} du_1 \dots du_k, \end{aligned}$$

and so,

$$(\Delta_L - 1)\hat{\phi} = -\Delta_L \phi.$$

(Q.E.D.)

## REFERENCES

- [1] Gelfand, I.M. and Vilenkin, N. Ya.: Generalized Functions, Vol. 4, English translation, Academic Press, New York (1964).
- [2] Gross, L.: Abstract Wiener Spaces, Proc. 5th. Berkeley Sym. Math. Stat. Prob. 2 (1965), 31-42.
- [3] Hida, T.: Analysis of Brownian Functionals, Carleton Math. Lecture Notes, No. 13, Carleton Univ. Ottawa, 1975.
- [4] Hida, T.: Generalized Multiple Wiener Integrals, Proc. Japan Acad. Ser. A Math. Sci. 54A (1978), 55-58.
- [5] Hida, T.: Brownian Motion, Application of Math. Vol. 11, Springer-Verlag, Heidelberg, Berlin, New York, 1980.
- [6] Hida, T.: Brownian Motion and its Functionals, Ricerche di Matematica, Vol. XXXIV, fasc. 1° (1985), 183-222.
- [7] Hida, T. and Saito, K.: White Noise Analysis and the Levy Laplacian, preprint (1986).
- [8] Ito, K.: Multiple Wiener Integrals, J. Math. Soc. Japan 3 (1951), 157-169.
- [9] Kubo, I. and Takenaka, S.: Calculus on Gaussian White Noise, I: Proc. Japan Acad. 56A (1980), 376-380, II: ibid. 56A (1980), 411-416, III: ibid. 57A (1981), 433-437, IV: ibid. 58A (1982), 186-189.
- [10] Kuo, H.-H.: Integration by Parts for Abstract Wiener Measures, Duke Math. J. 41 (1974), 373-379.
- [11] Kuo, H.-H.: Gaussian Measures in Banach Space, Lecture Notes in Math. Vol. 463, Springer-Verlag, Heidelberg, Berlin, New York, 1975.
- [12] Kuo, H.-H.: Integration in Banach Spaces, Notes in Banach Spaces, Ed. H. Elton Lacey, Univ. of Texas Press (1980), 1-38.

- [13] Kuo, H.-H.: On Fourier Transform of Generalized Brownian Functionals, *Journal of Multivariate Analysis* 12, 415-431 (1982).
- [14] Kuo, H.-H.: Brownian Functionals and Applications, *Acta Appl. Math.* Vol. 1 (1983), 175-188.
- [15] Kuo, H.-H.: On Laplacian Operators of Generalized Brownian Functionals, *Lecture Notes in Math.* Vol. 1203, Springer-Verlag, Berlin, Heidelberg, New York (1985), 119-128.
- [16] Piech, M.A.: The Ornstein-Uhlenbeck Semigroup in an Infinite Dimensional  $L^2$ -setting, *J. Func. Anal.* 18 (1975), 271-285.
- [17] Saito, K.: Ito's Formula and Levy Laplacian, preprint (1986).

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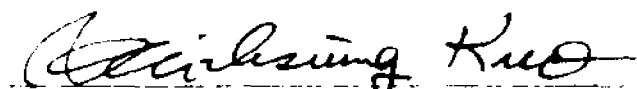
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Youngsook Lee SHIM


Major Field: Mathematics

Title of Dissertation: Abstract Wiener Space Approach to Hida Calculus

Approved:

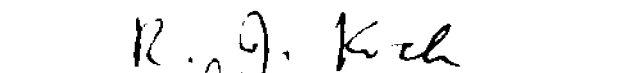


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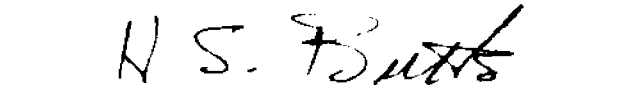


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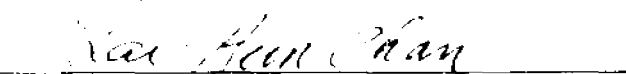
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